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The value of a merger and its optimal timing*

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Abstract

In this paper, we study a firm's merger strategy. When two firms merge, there are two types of transaction costs: fixed and proportional. To study the firm's merger strategy, we formulate the problem faced by the newly merged firm's management as an optimal stopping problem. Then, we derive the optimal merger strategy; i.e., we find the optimal value of the merger option. We also show that the optimal strategy is unique. Furthermore, we illustrate numerical examples and undertake a comparative static analysis of the merger option.

Keywords: merger; synergy effects; optimal stopping; variational inequalities; transaction costs
JEL classification: G34

1 Introduction

Mergers have become an interesting research topic in corporate finance. A merger is defined as an investment transaction for transferring corporate control by acquiring shares in another firm. It occurs when the managers of both firms believe that integrating their businesses will generate a higher market value than operating them separately. It is also an attempt by the managers to maximize the value of both firms by achieving synergies from business integration; i.e., managers seek to maximize stockholders' value. We describe the sources of synergy effects from mergers in Section 2. Firms' managers obtain a strategic advantage from a merger. Therefore, a firm is justified in merging only when the acquisition of some or all of an existing company's assets generates more growth opportunities than investing in its own equipment or infrastructure. In other words, a merger can increase the market value of a firm only when it produces new economic advantages.

Decision-making is complicated for the managers of both firms. The managers of the acquiring firm decide to execute a merger if, based on all the information available at the time,

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the merger will increase the market value of the acquiring firm. On the other hand, the managers of the acquired firm accept being overtaken only if they believe that this will increase the market value of their firm. Furthermore, the stockholder value of the acquired firm will rise if its managers accept a merger offer to integrate with the acquirer rather than remain a separate business. Therefore, a merger is an investment transaction that, in an uncertain environment, may generate more stockholder value for both merging companies than if they remain separate businesses.

Mergers with economic implications sometimes fail and generate worse outcomes because the managers cannot deal with the complicated task of integrating two companies with different production systems and corporate cultures. In such a case, the shareholder value following the merger is lower than the sum of the company's shareholder values before the merger. In this case, the investment in the merger is at least partly irreversible. Thus, in an uncertain environment, a merger is an investment transaction that can have irreversible consequences. However, both sides have the right to postpone the final decision between the time of the announcement of the merger and the time of its completion or dissolution.

In this paper, a merger is assumed to be undertaken through the stock exchange. Managers of each firm i ($i = 1, 2$) consider the amalgamation of both firms in order to maximize the value of the firms. In this context, we assume that a decision-making management, which is elected from the managers of firms 1 and 2, aims to maximize the value of the firm that results from the merger. To explain how this is achieved, we introduce the concept of the synergy effects of a merger. These effects arise from any source of value-creating efficiency that is generated by combining the assets, operations, and financial structures of the two firms in a merger. Then, the decision-making management's problem is to choose the timing of the merger to maximize its synergy effects. We formulate the manager's problem as an optimal stopping problem.

Related papers are as follows. Margrabe (1978), Carr (1995), and Morellec and Zhdanov (2005) investigate takeovers as exchange options. Margrabe (1978)'s study was the first to formulate takeovers as exchange options. Carr (1995) developed a general formula for valuing American exchange options on dividend-paying assets. In particular, Morellec and Zhdanov (2005) determine the optimal timing of a takeover by formulating takeover decisions as option exercise games. Lambrecht (2004) also investigates the timing of mergers. We follow Hu and Øksendal (1998) and Dupuis and Wang (2002) by formulating the manager's problem as an optimal stopping problem. Hu and Øksendal (1998) extend McDonald and Siegel (1986) analyze an investment timing problem by using optimal stopping. Dupuis and Wang (2002) study an optimal stopping problem by using the exogenous Poisson jump process to represent the signal process.

The structure of the paper is as follows. In Section 2, we formulate the decision-making manager's problem as an optimal stopping problem. In the subsequent section, we solve the manager's problem and show that a strategy, which is induced by variational inequalities, is optimal. In Section 4, we show that a candidate function of the value function of the manager's problem is a solution to the variational inequalities. That is, the candidate function is equal to the value function, and the strategy induced the variational inequalities is optimal. In Section 5, we present illustrative numerical examples and the results of comparative static analysis. Section 6 concludes the paper.

2 The Model of Merger Strategy

In this section, we develop the model of the firm's merger strategy. In this paper, the merger is assumed to be undertaken through the stock exchange. The managers of each firm i ($i = 1, 2$) consider amalgamating the firms to maximize their value. In this context, we assume that the decision-making management aims to maximize the value of the firm that results from the merger. We also assume that the two firms have no debt financing or have fixed amounts of debt financing over time. This allows us to focus on the market value of the stocks issued by the two firms. We assume that no large shareholders influence the management's merger decision.

In general, when a firm merges, it incurs transaction costs such as legal fees, fees paid to investment banks and other merger promoters, and the costs of restructuring and integrating the two firms. In this paper, we consider two types of transaction costs. Fixed transaction costs are the costs of setting up a merger deal. Variable transaction costs include, e.g., contingent fees.

The stock market exhibits a semi-strong form of efficiency. That is, market prices not only reflect past prices, but also reflect other published information, such as announcements about earnings and dividends, forecasts of corporate earnings, changes in accounting practices, and mergers. This information is rapidly and accurately reflected in the stock prices.

We assume that the market value of firm i, $S^i(t)$, is represented by the following stochastic differential equation:

$$\mathrm{d}S^i(t) = \alpha_i S^i(t)\mathrm{d}t + \sigma_i S^i(t)\mathrm{d}Z^i(t), \quad S_0^i \ (> 0), \tag{2.1}$$

where α_i ($\in \mathbb{R}$) is the expected growth rate of firm i. σ_i (> 0) is the standard deviation of the expected growth rate of firm i. Let $\mu_i = \alpha_i + \delta_i$ be the total expected rate of return of firm i, where δ_i is the expected dividend rate of firm i. We assume that the each parameter is constant. $Z(t) = (Z^1(t), Z^2(t))$ is a two-dimensional standard Brownian motion given on a filtered probability space, $(\Omega, \mathcal{F}^S, \mathbb{P}; \{\mathcal{F}^S(t)\}_{t\geq 0})$, and it satisfies the usual conditions.[1]. In this context, $\mathcal{F}(t)$ is generated by $Z(t)$ in $\mathbb{R}$; i.e., $\mathcal{F}^S(t) = \sigma(Z(s), s \leq t)$. The correlation between $Z^1(t)$ and $Z^2(t)$ is ρ_{12}. We assume that μ_i, σ_i, and ρ_{12} are constant.

Next, we discuss the source of synergy effects from mergers. One of the most important empirical questions in merger research is whether the combined return to two firms is positive or negative. In other words, are mergers positive net present value investments? Merger theories based on synergy predict that the combined return from a merger is positive. Most empirical analysis of the combined returns in mergers has employed event study methods. The evidence reviewed by Jensen and Runback (1983) indicates that mergers create wealth. Bradley, Desai and Kim (1988) conclude that "Successful tender offers generate synergistic gains and lead to a more efficient allocation of corporate resources" (page 13). Berkovitch and Narayanan (1993) conclude that synergy is the dominant force in mergers and takeovers. Mulherin and Boone (2000) find that the magnitude of the combined return is related directly to the relative size of the takeover event and conclude that their results "are consistent with the synergistic theory of the firm" (page 135). Andrade, Mitchell and Stafford (2001) conclude that "mergers create value on behalf of the shareholders of combined firms" (page 112).

Suppose that two firms, firms 1 and 2, are contemplating merging.[2] The difference between

[1] See, for example, Karatzas and Shreve (1991).

[2] In this context, the firms are assumed to be listed. It is reasonable that, for listed firms, the value of each firm can be determined by observing the market value of the outstanding securities.

the value of the merged firm and the sum of the values of the two firms as separate entities is the synergy from the merger:

$$\begin{aligned}\text{Synergy effects} =& \text{the value of the merged firm} \\ &- (\text{the value of firm } 1 + \text{the value of firm } 2).\end{aligned} \tag{2.2}$$

Synergy effects arise from any source of value-creating efficiency that is generated by combining the assets, operations, and financial structures of two firms in a merger. The two types of synergy effects that have been recognized in the literature are operating synergy effects and financial synergy effects. Operating synergy effects arise if the merger results in improvements in any business function.[3] Sources of operating synergy effects can be categorized into four types: (i) greater economies of scale; (ii) increased pricing power; (iii) combination of different functional strengths; and (iv) higher growth in new or existing markets. Operating synergy effects can influence margins and growth, and thereby the value of the firms involved in the merger.

Financial synergy effects are generated in a merger if some aspect of the financial configuration of the merged firm causes its market value to be greater than the sum of the market values of the separate firms. With financial synergy effects, the payoff can take the form of either higher cash flows or a lower cost of capital. These payoffs may arise from the following: (i) a merger between a firm with excess cash, or cash slack, and a firm with high-return project opportunities; (ii) debt capacity; and (iii) tax benefits from the use of net operating losses, unused debt capacity and/or surplus funds. Therefore, many mergers generate synergy effects. The more important issues are whether these synergy effects can be valued and, if they can, how.

Consider a decision on whether two firms should merge. Both management teams decide to merge the two firms into one only if the market value after merging is greater than the combined market values without a merger. This implies that the value of both firms combined before merging is the threshold for the merger. We denote this threshold by $X(t)$, which is a portfolio that consists of $S^1(t)$ and $S^2(t)$. Then, we can obtain the dynamics of $X = \{X(t)\}_{t\geq 0}$ by using (2.1). The total expected rate of return on this portfolio, μ_X, is given by

$$\begin{aligned}\mu_X &= w_1\mu_1 + w_2\mu_2 \\ &= w_1(\alpha_1 + \delta_1) + w_2(\alpha_2 + \delta_2) \\ &= \alpha_X + \delta_X,\end{aligned} \tag{2.3}$$

where $w_1 = S^1(0)/(S^1(0) + S^2(0))$, $w_2 = S^2(0)/(S^1(0) + S^2(0))$, $\alpha_X = w_1\alpha_1 + w_2\alpha_2$, and $\delta_X = w_1\delta_1 + w_2\delta_2$. For simplicity, we assume that w_i $(i = 1, 2)$ is constant. The volatility of the portfolio, σ_X, is given by

$$\sigma_X = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}. \tag{2.4}$$

Then, the dynamics of X are given by

$$\mathrm{d}X(t) = \mu_X X(t)\mathrm{d}t + \sigma_X X(t)\mathrm{d}Z^X(t), \quad X(0) = x\ (> 0), \tag{2.5}$$

where $Z^X(t)$ is also a standard Brownian motion given on a filtered probability space, $(\Omega, \mathcal{F}^S, \mathbb{P}; \{\mathcal{F}^S(t)\}_{t\geq 0})$.

[3] For example, there may be improvements in management, labor costs, production and distribution, resource acquisition and allocation, and market power.

Let $U(t)$ be the market value of the combined firm. We assume that $U = \{U(t)\}_{t\geq 0}$ is governed by the following stochastic differential equation:

$$\mathrm{d}U(t) = \alpha_U U(t)\mathrm{d}t + \sigma_U U(t)\mathrm{d}Z^U(t), \quad U(0) = u \ (> 0), \tag{2.6}$$

where α_U $(\in \mathbb{R})$ is the expected growth rate of the combined firm, which is defined by $\alpha_U = \gamma_\alpha + \alpha_X$. σ_U (> 0) is the standard deviation of the expected growth rate of the combined firm, defined by $\eta\sigma_X$. γ_α (> 0) is the synergy effect parameter for the expected growth rate. η (> 0) is the synergy effect parameter for the standard deviation. $Z^U(t)$ is a standard Brownian motion on a filtered probability space, $(\Omega, \mathcal{F}^U, \mathbb{P}; \{\mathcal{F}^U(t)\}_{t\geq 0})$. Let $\mathcal{F}(t)$ be defined by $\mathcal{F}(t) := \mathcal{F}^S(t) \vee \mathcal{F}^U(t)$. Let $\mu_U = \alpha_U + \delta_U$ be the total expected rate of return of the combined firm, where δ_U is the expected dividend rate for the combined firm, which is defined by $\gamma_\delta + \delta_X$. γ_δ (> 0) is the synergy effects parameter for the dividend rate. For simplicity, we assume that $\gamma := \gamma_\alpha = \gamma_\delta$. It transpires that $\mu_U = 2\gamma + \mu_X$.

Given the assumption of a semi-strong form of market efficiency, at time $t = 0-$, before the firms announce the merger, the merged firm's market value, $U(0-)$, is equal to the sum of two firms' market values, $S^1(0-) + S^2(0-)$. This implies that there is no leakage of information. However, at time $t = 0$, when the firms announce the merger, one would expect the merger announcement to be immediately reflected in the stock prices of two firms and, hence, in the two firms' market values. Therefore, $U(t)$ and $X(t)$ fluctuate separately over time after the merger is announced. (See figure 1.)

We can now formulate the model of the merger option to obtain the synergy effects. The combined firm's expected discounted synergy effects, $J(u, x; \tau)$, are defined by

$$J(u, x; \tau) = \mathbb{E}\left[e^{-r\tau}(U(\tau) - k_1 X(\tau)) - \hat{k}_0\right], \tag{2.7}$$

where r is a discount rate, τ $(< \infty)$ is the time of the merger, $\hat{k}_0$ is the fixed transaction cost, and k_1 (> 1) is the proportional transaction cost parameter. τ is an $\{\mathcal{F}_t\}_{t\geq 0}$-stopping time. For example, $\hat{k}_0$ represents the setup cost and $(k_1 - 1)$ represents the contingent fee.

Let $Y(t) = U(t)/X(t)$ be the proportion of the combined firm's value to the portfolio of the two firms. Then, the dynamics of $Y = \{Y(t)\}_{t\geq 0}$ are governed by

$$\mathrm{d}Y(t) = \mu_Y Y(t)\mathrm{d}t + \sigma_U Y(t)\mathrm{d}Z^U(t) - \sigma_X Y(t)\mathrm{d}Z^X(t), \quad Y(0) = y \ (> 0), \tag{2.8}$$

where $\mu_Y = \mu_U - \mu_X + \sigma_X^2 - \rho_{UX}\sigma_U\sigma_X = 2\gamma + (1 - \rho_{UX}\eta)\sigma_X^2$. Then, the combined firm's expected discounted synergy effects, $J(u, x; \tau)$, can be rewritten as

$$J(y; \tau) = \mathbb{E}\left[e^{-r\tau}(Y(\tau) - k_1) - k_0\right], \tag{2.9}$$

where $k_0 = \hat{k}_0/X(0)$. Let $\mathcal{R} \subset \mathbb{R}_+$ be a domain over which both firms are active. Then, we define the bankruptcy time, $\tau_{\mathcal{R}}$, as

$$\tau_{\mathcal{R}} = \inf\{t > 0; Y(t) \notin \mathcal{R}\}. \tag{2.10}$$

Therefore, the decision-making management's problem is to choose the merger time, τ, to maximize the synergy effects from the merger:

$$V(y) = \sup_{\tau\in\mathcal{T}} J(y; \tau), \tag{2.11}$$

where V is the value function and $\mathcal{T}$ is the set of all admissible merger times, $\tau < \tau_{\mathcal{R}}$. In this context, we assume that

$$r - \mu_Y > 0. \tag{AS.1}$$

3 Verification Theorem

In the previous section, we formulated the firms' problem as an optimal stopping problem. We solve this problem by the variational inequalities (VI). Then, we show that a solution of the VI is the value function. This is the verification theorem.

If a function, ϕ, is a twice continuously differentiable function with respect to Y, $C^2(\mathcal{R})$, then an operator, $\mathcal{L}$, is defined by

$$\mathcal{L}\phi(y) = \frac{1}{2}\lambda y^2 \phi''(y) + \mu_Y y \phi'(y) - r\phi(y), \tag{3.1}$$

where $\lambda = \sigma_U^2 - 2\rho_{UX}\sigma_U\sigma_X + \sigma_X^2 = (\eta^2 - 2\rho_{UX}\eta + 1)\sigma_X^2$. Let $\mathcal{C}$ be a continuation region, that is, for $y \in \mathcal{C}$, the firm does not exercise the merger option. When the firm merges, the process shift X to U. Then, for $y \in \partial\mathcal{C}$, ϕ is not $C^2(\mathcal{R})$, where $\partial\mathcal{C}$ is the boundary of the region $\mathcal{C}$. Suppose that $Y(t)$ spends no time on $\partial\mathcal{C}$ a.s. That is,

$$\mathbb{E}\left[\int_0^\infty Y(t) 1_{\{Y(t)\in\partial\mathcal{C}\}} \mathrm{d}t\right] = 0, \tag{3.2}$$

where $1_{\{\cdot\}}$ is an indicator function. $1_{\{Y(t)\in\partial\mathcal{C}\}}$ means that if $Y(t) \in \partial\mathcal{C}$, then $1_{\{Y(t)\in\partial\mathcal{C}\}} = 1$. However, if $Y(t) \notin \partial\mathcal{C}$, then $1_{\{Y(t)\in\partial\mathcal{C}\}} = 0$. Hence, we assume that $\phi \in C^1(\mathcal{R})$, $\phi \in C^2(\mathcal{R}\backslash\partial\mathcal{C})$, and the second derivatives of ϕ are locally bounded near $\partial\mathcal{C}$. Furthermore, we assume that $\partial\mathcal{C}$ is a Lipschitz surface.

Next, we define the VI.

Definition 3.1 (VI). *The following relations are the VI for the firm's problem (2.11):*

$$\mathcal{L}\phi(y) \leq 0; \tag{3.3}$$

$$\phi(y) \geq (y - k_1) - k_0; \tag{3.4}$$

$$[\mathcal{L}\phi(y)][((y - k_1) - k_0) - \phi(y)] = 0. \tag{3.5}$$

(3.5) is the complementary condition and can be rewritten as follows:

$$\mathcal{L}\phi(y) = 0, \quad y \in \mathcal{C} \tag{3.6}$$

and

$$((y - k_1) - k_0) - \phi(y) = 0, \quad y \notin \mathcal{C}, \tag{3.7}$$

where $\mathcal{C}$ is the continuation region defined by

$$\mathcal{C} := \{y; \phi(y) > ((y - k_1) - k_0) \text{ and } \mathcal{L}\phi(y) = 0\}. \tag{3.8}$$

Then, it follows from the VI that the merger time, $\tau_{\mathcal{C}}$, is defined by

$$\tau_{\mathcal{C}} = \inf\{t \geq 0; y \notin \mathcal{C}\}. \tag{3.9}$$

The following equality, which is termed the *Dynkin formula*, is used in the proof of Theorem 3.1:

$$\mathbb{E}[e^{-r\tau}\phi(Y(\tau))] = \phi(y) + \mathbb{E}\left[\int_0^\tau e^{-rt}\mathcal{L}\phi(Y(t))\mathrm{d}t\right]. \tag{3.10}$$

See Øksendal (2003) for details of the Dynkin formula.

We can now prove that a strategy induced by the VI is an optimal merger strategy. The following theorem is the well-known verification theorem. See, e.g., Theorem 10.4.1 in Øksendal (2003).

Theorem 3.1. *Suppose that (AS.1) holds.*

(I) Let ϕ be a solution of the VI (3.3)–(3.5) that satisfies the following:

$$\begin{aligned}&\text{the family } \{\phi(Y(\tau))\}_{\tau\in\hat{\mathcal{T}}} \text{ is uniformly integrable with respect to } \mathbb{P},\\ &\text{where } \hat{\mathcal{T}} \text{ is the set of all bounded stopping times } \tau \leq \tau_{\mathcal{C}}.\end{aligned} \tag{3.11}$$

Then, we obtain the following:

$$\phi(y) \geq V(y). \tag{3.12}$$

(II) From (3.6), we obtain

$$\mathcal{L}\phi(y) = 0, \quad y \in \mathcal{C}. \tag{3.13}$$

Furthermore, the merger time is given by (3.9). Then, we find that the function ϕ is equal to the value function, V:

$$\phi(y) = V(y). \tag{3.14}$$

In addition, $\tau_{\mathcal{C}}$ is optimal.

Proof. (I) Since $\phi \in C^1(\mathcal{R})$, $\partial\mathcal{C}$ is a Lipschitz surface, $\phi \in C^2(\mathcal{R}\backslash\partial\mathcal{C})$ and because the second derivatives of ϕ are locally bounded near $\partial\mathcal{C}$, we can find a sequence of functions, $\phi_j \in C^2(\mathcal{R}) \cap C(cl\mathcal{R})$, $j = 1, 2, \cdots$, such that

$$\begin{aligned}&\phi_j \to \phi \text{ uniformly on compact subsets of } cl\mathcal{R}, \ as\ j \to \infty;\\ &\mathcal{L}\phi_j \to \mathcal{L}\phi \text{ uniformly on compact subset of } \mathcal{R}\backslash\partial\mathcal{C}, \ as\ j \to \infty;\\ &\{\mathcal{L}\phi_j\}_{j=1}^{\infty} \text{ is locally bounded on } \mathcal{R},\end{aligned} \tag{3.15}$$

where $cl\mathcal{R}$ is the closure of the region $\mathcal{R}$.

Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of bounded open sets such that $\mathcal{R} = \bigcup_{n=1}^{\infty} G_n$. Let T_n be defined by $T_n = \min[n, \inf\{t > 0; U(t) \notin G_n\}]$, and let $\tau \leq \tau_{\mathcal{R}}$ be a stopping time. Then, from (3.10), it follows that, for $y, x \in \mathcal{R}$

$$\mathbb{E}[e^{-r(\tau\wedge T_n)}\phi_j(Y(\tau \wedge T_n))] = \phi_j(y) + \mathbb{E}\left[\int_0^{\tau\wedge T_n} e^{-rt}\mathcal{L}\phi_j(Y(t))\mathrm{d}t\right]. \tag{3.16}$$

Taking $\lim_{s\to\infty}$ in (3.16) and using (3.15) and (3.2) yields

$$\begin{aligned}\phi(y) &= \lim_{j\to\infty} \mathbb{E}\left[-\int_0^{\tau\wedge T_n} e^{-rt}\mathcal{L}\phi_j(Y(t)))\mathrm{d}t + e^{-r(\tau\wedge T_n)}\phi_j(Y(\tau\wedge T_n))\right] \\ &= \mathbb{E}\left[-\int_0^{\tau\wedge T_n} e^{-rt}\mathcal{L}\phi(Y(t))\mathrm{d}t + e^{-r(\tau\wedge T_n)}\phi(Y(\tau\wedge T_n))\right]\end{aligned} \tag{3.17}$$

From (3.3), we obtain

$$\phi(y) \geq \mathbb{E}\left[e^{-r(\tau\wedge T_n)}\phi(Y(\tau\wedge T_n))\right]. \tag{3.18}$$

It follows from the Fatou's Lemma that

$$\phi(y) \geq \liminf_{n\to\infty} \mathbb{E}\left[e^{-r(\tau\wedge T_n)}\phi(Y(\tau\wedge T_n))\right] \geq \left[e^{-r\tau}\phi(Y(\tau))\right]. \tag{3.19}$$

Since $\tau \leq \tau_{\mathcal{R}}$ is arbitrary, we have

$$\phi(y) \geq V(y), \quad y \in \mathcal{R}. \tag{3.20}$$

Hence, (3.12) is verified.

(II) First, suppose that $u \notin \mathcal{C}$. It follows from (3.7) that

$$\phi(y) = [(y - k_1) - k_0] \leq V(y). \tag{3.21}$$

Then, given (3.20), it follows that

$$\phi(y) = V(y), \quad y \notin \mathcal{C}; \tag{3.22}$$

$$\tau^* = 0 \text{ is optimal}, \quad y \notin \mathcal{C}. \tag{3.23}$$

Next, suppose that $u \in \mathcal{C}$. Let $\{\mathcal{C}_m\}_{m=1}^{\infty}$ be an increasing sequence of open sets $\mathcal{C}_m$ such that $cl\mathcal{C}_m \subset \mathcal{C}$, $cl\mathcal{C}_m$ is compact and $\mathcal{C} = \bigcup_{m=1}^{\infty}\mathcal{C}_m$. Let τ_m be defined by $\tau_m = \inf\{t > 0; U(t) \notin \mathcal{C}_m\}$, $m = 1, 2, \cdots$. Then, from (3.10), it follows for $u \in \mathcal{C}_m$ that

$$\begin{aligned}\phi(y) &= \lim_{j\to\infty} \phi_j(y) \\ &= \lim_{j\to\infty} \mathbb{E}\left[-\int_0^{\tau_m\wedge T_n} e^{-rt}\mathcal{L}\phi_j(Y(t))\mathrm{d}t + e^{-r(\tau\wedge T_n)}\phi_j(Y(\tau_m\wedge T_n))\right] \\ &= \mathbb{E}\left[-\int_0^{\tau_m\wedge T_n} e^{-rt}\mathcal{L}\phi(Y(t))\mathrm{d}t + e^{-r(\tau_m\wedge T_n)}\phi(Y(\tau_m\wedge T_n))\right] \\ &= \mathbb{E}\left[e^{-r(\tau_m\wedge T_n)}\phi(Y(\tau_m\wedge T_n))\right].\end{aligned} \tag{3.24}$$

Taking $\lim_{n\to\infty,m\to\infty}$ in (3.24) and using (3.9) and (3.2) yields

$$\begin{aligned}\phi(y) &= \lim_{n\to\infty,m\to\infty} \mathbb{E}\left[e^{-r(\tau_m\wedge T_n)}\phi(Y(\tau_m\wedge T_n))\right] \\ &= \mathbb{E}\left[e^{-r\tau_{\mathcal{C}}}\phi(Y(\tau_{\mathcal{C}}))\right] \\ &= J(y;\tau_{\mathcal{C}}) \leq V(y).\end{aligned} \tag{3.25}$$

Then, combining (3.20) and (3.25) yields

$$\phi(y) \geq V(y) \geq J(y; \tau_{\mathcal{C}}) = \phi(y). \tag{3.26}$$

Then, we have the following results:

$$\phi(y) = V(y), \quad y \in \mathcal{C}; \tag{3.27}$$

$$\tau^* = \tau_{\mathcal{C}} \text{ is optimal}, \quad y \in \mathcal{C}. \tag{3.28}$$

Therefore, from (3.22), (3.23), (3.27), and (3.28), we conclude that the function ϕ is equal to the value function:

$$\phi(y) = V(y), \quad y \in \mathcal{R}. \tag{3.29}$$

Furthermore, the following stopping time τ^* is optimal:

$$\tau^* = \begin{cases} 0, & y \notin \mathcal{C}, \\ \tau_{\mathcal{C}}, & y \in \mathcal{C}. \end{cases} \tag{3.30}$$

Therefore, $\tau_{\mathcal{C}}$ is optimal. This completes the proof.

□

4 Optimal Merger Strategy

In this section, we investigate whether the candidate function, ϕ, is a solution to the VI. From the formulation of the firm's problem, we guess the optimal merger strategy as follows. If the process Y exceeds a threshold, y^*, the firm exercises the merger option, and otherwise it does not. Thus, the optimal merger strategy is given by

$$\tau^* = \inf\{t > 0; Y(t) \geq y^*\}. \tag{4.1}$$

For $y < y^*$, it follows from (3.6) that

$$\mathcal{L}\phi(y) = 0. \tag{4.2}$$

The standard approach to solving ordinary differential equations implies that the general solution to (4.2) is of the following form:

$$\phi(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2}, \tag{4.3}$$

where A_1 and A_2 are constants to be determined, and β_1 and β_2 are the solutions to the following characteristic equation:

$$\frac{1}{2}\lambda\beta(\beta - 1) + \mu_Y \beta - r = 0. \tag{4.4}$$

Then, β_1 and β_2 are

$$\beta_1 = \left(\frac{1}{2} - \frac{\mu_Y}{\lambda}\right) + \left[\left(\frac{\mu_Y}{\lambda} - \frac{1}{2}\right)^2 + \frac{2r}{\lambda}\right]^{1/2}; \qquad \beta_2 = \left(\frac{1}{2} - \frac{\mu_Y}{\lambda}\right) - \left[\left(\frac{\mu_Y}{\lambda} - \frac{1}{2}\right)^2 + \frac{2r}{\lambda}\right]^{1/2}. \tag{4.5}$$

Since for $y \in \mathcal{R}$ the firm is active, we obtain that

$$\lim_{y \to 0} V(y) = 0. \tag{4.6}$$

Since $\beta_1 > 1$ and $\beta_2 < 0$, it follows from (4.6) that $A_2 = 0$. Then, we have

$$\phi(y) = A_1 y^{\beta_1}, \quad y < y^*. \tag{4.7}$$

The two unknown parameters A_1 and y^* are determined by the following simultaneous equations:

$$\phi(y^*) = y^* - k_1 - k_0; \tag{4.8}$$

$$\phi'(y^*) = 1. \tag{4.9}$$

(4.8) and (4.9) are the well-known value matching and smooth pasting conditions, respectively. These conditions yield

$$y^* = \left(\frac{\beta_1}{\beta_1 - 1}\right)(k_1 + k_0) \tag{4.10}$$

$$A_1 = \frac{1}{\beta_1}\left(\frac{\beta_1}{\beta_1 - 1}\right)^{1-\beta_1}(k_1 + k_0)^{1-\beta_1}. \tag{4.11}$$

Let us define the function $\varphi(y)$ as follows:

$$\varphi(y) = \begin{cases} \phi(y), & y < y^*, \\ (y - k_1) - k_0, & y \geq y^*. \end{cases} \tag{4.12}$$

Then, we obtain the following proposition.

Proposition 4.1. *Suppose that (AS.1) holds. Furthermore, we assume that*

$$\beta_1 \mu_Y > r. \tag{AS.2}$$

Then, the function φ satisfies the VI: (3.3) – (3.5).

Proof. First, we show that φ satisfies (3.3) for $y \in \mathcal{R}$. For this, we divide $\mathcal{R}$ into two intervals, $y < y^*$ and $y \geq y^*$. For $y < y^*$, we have $\varphi = \phi$ from (4.12). Thus, it follows from (4.2) that $\mathcal{L}\varphi(y) = 0$. For $y \geq y^*$, we have $\varphi = (y - k_1) - k_0$ from (4.12). Given that $\varphi''(y) = 0$ and $\varphi'(y) = 1$, we have

$$\begin{aligned} \mathcal{L}\varphi(y) &= \mu_Y y - r[(y - k_1) - k_0] \\ &= (\mu_Y - r)y + r(k_1 + k_0) \end{aligned} \tag{4.13}$$

Because y^* is the smallest value in $y \geq y^*$, it is sufficient to show that (4.13) is negative at $y = y^*$. Equivalently, it is sufficient to show that

$$\left(\frac{\beta_1}{\beta_1 - 1}\right) < \left(\frac{r}{r - \mu_Y}\right). \tag{4.14}$$

Given (AS.1) and (AS.2), (4.14) holds. Then we can show that φ satisfies (3.3) for all $y \in \mathcal{R}$.

Second, we show that φ satisfies (3.4). Let $\Phi(y)$ be defined by

$$\Phi(y) = [(y - k_1) - k_0] - A_1 y^{\beta_1}. \tag{4.15}$$

This implies the following:

$$\lim_{y \to 0} \Phi(y) = -(k_1 + k_0) < 0; \tag{4.16}$$

$$\Phi'(y) = 1 - \beta_1 A_1 y^{\beta_1 - 1}; \tag{4.17}$$

$$\lim_{y \to 0} \Phi'(y) = 1; \qquad \lim_{y \to \infty} \Phi'(y) = -\infty. \tag{4.18}$$

It is clear from eqs. (4.10) and (4.11) that

$$\Phi'(y) \begin{cases} > 0, & y < y^*, \\ = 0, & y = y^*, \\ < 0, & y > y^*. \end{cases} \tag{4.19}$$

The unique zero point of this equation is $y = y^*$. For $y < y^*$, we have $\varphi = \phi$ from (4.12). For $y \geq y^*$, we have $\varphi = (y - k_1) - k_0$ from (4.12). This implies the following:

$$[(y - k_1) - k_0] - \varphi(y) = 0; \tag{4.20}$$

Thus, we have shown that φ satisfies (3.4) for all $y \in \mathcal{R}$.

It is clear that φ satisfies (3.5) for all $y \in \mathcal{R}$.

□

Theorem 3.1 and Proposition 4.1 imply the following theorem.

Theorem 4.1. *The function φ satisfies the VI: (3.3) – (3.5). Hence, $\varphi(y) = V(y)$ and the merger strategy implied by the VI is optimal. That is, τ^* defined by (3.30) is optimal. Furthermore, because Proposition 4.1 implies that the threshold y^* is unique, τ^* is also unique.*

5 Numerical Examples

In this section, we numerically evaluate β_1, A_1, and y^*. Hence, we obtain the value of the merger option, $A_1(y^*)^{\beta_1}$. To do this, we use the base case parameter values reported in Table 1. The results are reported in Table 2. We also vary the parameters by ±30% and report the associated changes in β_1, A_1, y^*, and $A_1(y^*)^{\beta_1}$. In addition, Figure 2 illustrates the function φ defined by (4.12). Table 2 illustrates the comparative statics.

Proposition 5.1. *Suppose that (AS.1) holds. The threshold, y^*, and the value of the merger option, $A_1(y^*)^{\beta_1}$, increase in the following: the standard deviation of the expected growth rate of firm i, σ_i; the constant correlation coefficient between Z^1 and Z^2, ρ_{12}; the synergy effects parameter for μ_X, γ; the fixed transaction cost, $\hat{k}_0$; and the proportional transaction cost parameter, k_1. If the constant correlation between Z^U and Z^X, ρ_{UX}, is negative, then y^* and $A_1(y^*)^{\beta_1}$ increase in the synergy effects parameter for σ_X, η. We find that y^* and $A_1(y^*)^{\beta_1}$ decrease in the following: the discount rate, r; the initial value of firm i, $S^i(0)$; and ρ_{UX}. If ρ_{UX} is positive, y^* and $A_1(y^*)^{\beta_1}$ decrease in η. In our formulation, the effects on y^* and $A_1(y^*)^{\beta_1}$ of the expected growth rate of firm i, α_i, and firm i's dividend rate, δ_i, reduce to γ. Thus, α_i and δ_i do not directly affect y^* and $A_1(y^*)^{\beta_1}$.*

Proposition 5.1 implies the following. Given the effect of σ_i, mergers are worthwhile for firms that face substantial uncertainty, because of the establishment of new firms, for instance. The predicted effects of $\hat{k}_0$ and k_1 imply that greater synergy effects from mergers are required for higher option values. The effect of ρ_{12} implies that option values are higher for two firms that operate in the same industry. The predicted effect of ρ_{UX} implies that a merger is more beneficial to poorly performing firms. Because $\mu_Y = 2\gamma + (1 - \rho_{UX}\eta)\sigma_X^2$, μ_Y increases in γ. μ_Y decreases (resp. increases) in η, if ρ_{UX} is positive (resp. negative).

6 Conclusions

In this paper, we considered a merger between two firms. In this context, we assumed that there is a decision-making management whose aim is to maximize the value of the newly merged firm as if it were managing the merged firm. When two firms merge, two types of transaction costs, fixed and proportional, are incurred. To analyze the firm's merger strategy, we formulated the decision-making management's problem as an optimal stopping problem. To solve the problem, we used the variational inequalities (VI). Then, we showed that a solution of the VI is the value function. That is, the candidate function is equal to the value function of the manager's problem, and the strategy implied by the VI is optimal. We derived the following optimal merger strategy. The firm exercises the merger option only if the process Y, which is the proportion of the combined firm's value to the portfolio of the two firms, exceeds the threshold y^*. We also showed that the optimal strategy is unique. Furthermore, we presented illustrative numerical examples and comparative statics.

We formulated the economic value of the integrated firm's synergy effects by assuming that the merger is conducted through the stock exchange based on a 'fair' ratio of the acquired firm's market value to that of an acquiring firm, w_2/w_1. In the context of the issue of 'pricing', because we assumed that the acquiring firm purchases the acquired firm's stock by using cash, stock, or other securities, the issue of a 'premium' (or discount) did not arise.

Our model could also be applied when a 'premium' is paid by the acquiring firm to the acquired firm's shareholders to execute a merger. Furthermore, our model suggests how the value of the synergy effects obtained from the merger of two firms might be divided between the shareholders of the acquiring firm and those of the acquired firm. When the acquiring firm purchases the acquired firm's stock at the 'fair' ratio, w_2/w_1, at the time of the merger or acquisition, the additional value generated by the merger could be allocated to the shareholders of the two firms in accordance with this ratio. When the actual ratio is higher than the 'fair' ratio, the acquiring firm pays a 'premium' to purchase the acquired firm's stock. When the actual ratio is below the 'fair' ratio, the acquiring firm purchases the acquired firm's stock at a discounted price. Therefore, we can apply our model to any acquisition by using the ratio, w_2/w_1.

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Figure 1: Paths of the processes U and X.

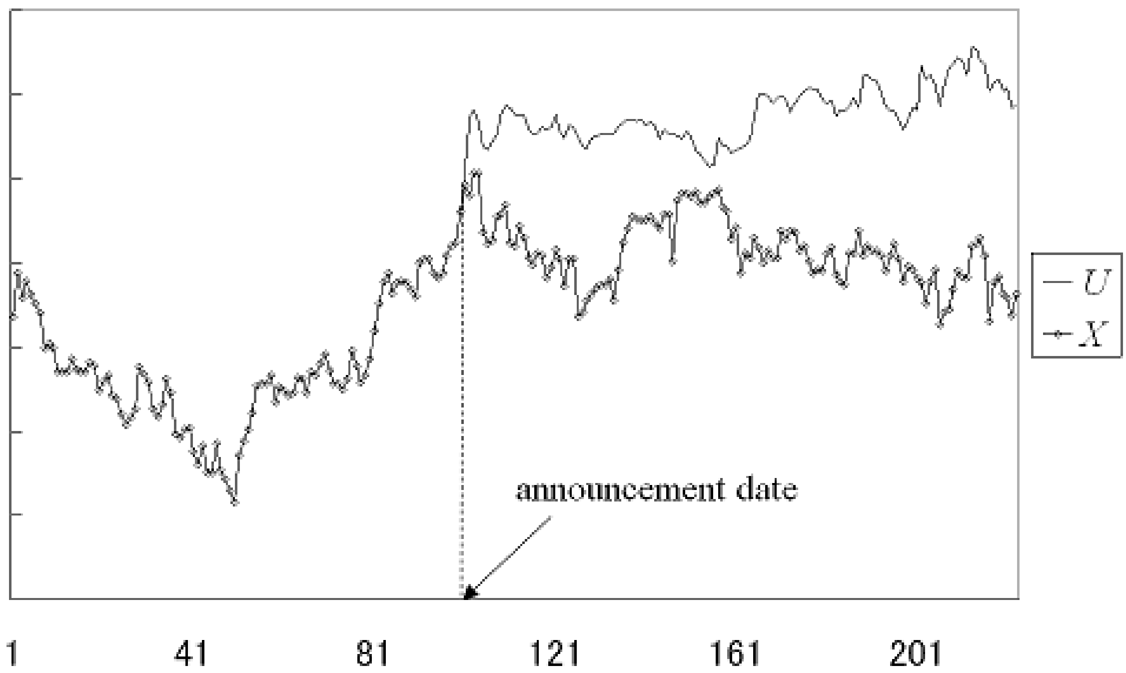

Figure 2: The value function, $\varphi(y)$

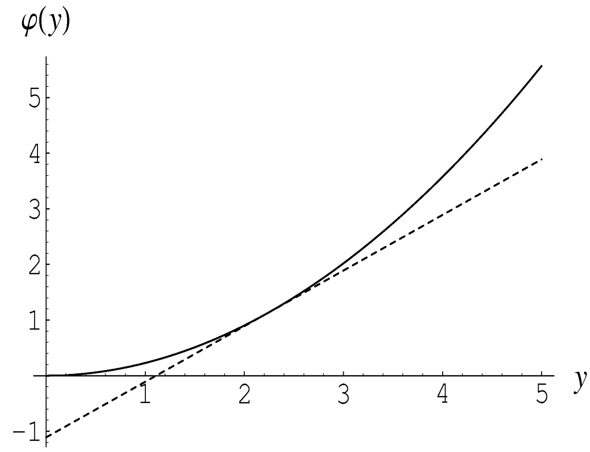

Table 1: Base case parameter values

Parameters	Values	
r	0.2	the discount rate
α_1	0.04	the expected growth rate of firm 1
α_2	0.05	the expected growth rate of firm 2
δ_1	0.01	the expected dividend rate of firm 1
δ_2	0.005	the expected dividend rate of firm 2
μ_1	0.05	$\mu_1 = \alpha_1 + \delta_1$
μ_2	0.055	$\mu_2 = \alpha_2 + \delta_2$
σ_1	0.2	the standard deviation of the expected growth rate of firm 1
σ_2	0.3	the standard deviation of the expected growth rate of firm 2
ρ_{12}	0.5	the constant correlation between Z^1 and Z^2
$S_1(0)$	60	the initial value of firm 1
$S_2(0)$	40	the initial value of firm 2
w_1	0.6	$w_1 = S^1(0)/(S^1(0) + S^2(0))$
w_2	0.4	$w_2 = S^2(0)/(S^1(0) + S^2(0))$
μ_X	0.052	$w_1\mu_1 + w_2\mu_2$
σ_X	0.0432	$w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}$
$X(0)$	100	$S_1(0) + S_2(0)$
γ	0.05	the synergy effects parameter for μ_X
η	1.05	the synergy effects parameter for σ_X
α_U	0.05	the expected growth rate of the combined firm: $\alpha_U = \gamma\alpha_X$
δ_U	0.008	the expected dividend rate of the combined firm: $\delta_U = \gamma\delta_U$
μ_U	0.152	the expected total growth rate of the combined firm: $\mu_U = 2\gamma + \mu_X$
σ_U	0.04536	the standard deviation of the expected growth rate of the combined firm: $\sigma_U = \eta\sigma_X$
ρ_{UX}	0.8	the constant correlation between Z^U and Z^X
μ_Y	0.10030	$\mu_Y = 2\gamma + (1 - \rho_{UX}\eta)\sigma_X^2$
λ	0.00079	$\lambda = (\eta^2 - 2\rho_{UX}\eta + 1)\sigma_X^2$
$\hat{k}_0$	1	the fixed transaction cost
k_0	0.01	$k_0 = \hat{k}_0/X(0)$
k_1	1.1	the proportional transaction cost parameter

Table 2: The results of numerical examples.

Panel A

	β_1	A_1	y^*	$A_1(y^*)^{\beta_1}$
Base case	1.9863	0.2277	2.2354	1.1254
$r*1.3$	2.5763	0.1518	1.8142	0.7042
$r*0.7$	1.3937	0.4187	3.9296	2.8196
$\alpha_1*1.3$	1.9863	0.2277	2.2354	1.1254
$\alpha_1*0.7$	1.9863	0.2277	2.2354	1.1254
$\delta_1*1.3$	1.9863	0.2277	2.2354	1.1254
$\delta_1*0.7$	1.9863	0.2277	2.2354	1.1254
$\sigma_1*1.3$	1.9761	0.2296	2.2472	1.1372
$\sigma_1*0.7$	1.9927	0.2265	2.2282	1.1182
$\rho_{12}*1.3$	1.9835	0.2282	2.2386	1.1286
$\rho_{12}*0.7$	1.9889	0.2272	2.2324	1.1224
$-\rho_{12}$	1.9985	0.2255	2.2217	1.1117
$-\rho_{12}*1.3$	1.9992	0.2254	2.2208	1.1108
$S^1(0)*1.3$	1.9875	0.2278	2.231	1.1225
$S^1(0)*0.7$	1.9839	0.2277	2.2425	1.1303
$\gamma*1.3$	1.5325	0.3516	3.1946	2.0846
$\gamma*0.7$	2.8163	0.1324	1.7211	0.6111
$\eta*1.3$	1.9909	0.2269	2.2302	1.1202
$\eta*0.7$	1.9782	0.2292	2.2447	1.1347
$\rho_{UX}=1$	2.0018	0.2249	2.218	1.1080
$\rho_{UX}*0.7$	1.9684	0.231	2.2562	1.1462
$-\rho_{UX}*0.7$	1.8923	0.2462	2.354	1.2440
$-\rho_{UX}$	1.8774	0.2494	2.3751	1.2651
$\hat{h}_0*1.3$	1.9863	0.2271	2.2414	1.1284
$\hat{h}_0*0.7$	1.9863	0.2283	2.2293	1.1223
$k_1*1.3$	1.9863	0.1761	2.8999	1.4599
$k_1*0.7$	1.9863	0.3225	1.5708	0.7908

Base case parameter values are used in Table 1.

Panel B

	β_1	A_1	y^*	$A_1(y^*)^{\beta_1}$
$\eta*1.3$	1.8532	0.2547	2.4110	1.3010
$\eta*0.7$	1.9003	0.2445	2.3429	1.2329

We assume that $\rho_{UX}=-0.8$. The other parameter values are those used for Table 1.