# Supplement to
# "Nonparametric Estimation and Testing on Discontinuity of Positive Supported Densities: A Kernel Truncation Approach"

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## A Technical proofs

### A.1 List of useful formulae

The formulae below are frequently used in the technical proofs.

***Stirling's formula.***

$$\Gamma(a+1)=\sqrt{2\pi}a^{a+1/2}\exp(-a)\left\{1+\frac{1}{12a}+O\left(a^{-2}\right)\right\} \text{ as } a\to\infty. \tag{A1}$$

***Recursive formulae on incomplete gamma functions.***

$$\gamma(a+1,z) = a\gamma(a,z)-z^{a}\exp(-z) \text{ for } a,z>0. \tag{A2}$$

$$\Gamma(a+1,z) = a\Gamma(a,z)+z^{a}\exp(-z) \text{ for } a,z>0. \tag{A3}$$

***Identity among gamma and incomplete gamma functions.***

$$\gamma(a,z)+\Gamma(a,z)=\Gamma(a) \text{ for } a,z>0. \tag{A4}$$

## A.2 Proof of Proposition 1

To save space, we only provide approximations to the bias and variance of $\hat{f}_-(c)$. Using (A3), (A4) and (A5) gives the results for $\hat{f}_+(c)$ in the same manner. The proof utilizes the following asymptotic expansion:

$$\frac{\gamma(a,a)}{\Gamma(a)} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{3a^{1/2}} + \frac{1}{540a^{3/2}} + O\left(a^{-5/2}\right)\right\} \text{ as } a \to \infty. \tag{A5}$$

This can be obtained by either letting $x \downarrow 0$ in equation (1) of Pagurova (1965) or putting $\eta = 0$ in equation (1.4) of Temme (1979). Then, putting $z = a$ in (A2) and then substituting (A1) and (A5), we have

$$\begin{aligned}\frac{\gamma(a+1,a)}{\Gamma(a+1)} &= \frac{\gamma(a,a)}{\Gamma(a)} - \frac{a^a \exp(-a)}{\Gamma(a+1)} \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(-\frac{2}{3}a^{-1/2} + \frac{23}{270}a^{-3/2}\right) + O\left(a^{-5/2}\right).\end{aligned} \tag{A6}$$

**Bias.** By the change of variable $v := u/b$,

$$E\left\{\hat{f}_-(c)\right\} = \int_0^c \frac{u^{c/b}\exp(-u/b)}{b^{c/b+1}\gamma(c/b+1,c/b)} f(u)\,du = \int_0^a f(bv)\left\{\frac{v^a \exp(-v)}{\gamma(a+1,a)}\right\} dv,$$

where $a := c/b$ and the object inside brackets of the right-hand side is a pdf on the interval $[0,a]$. Then, a second-order Taylor expansion of $f(bv)$ around $bv = c$ (from below) yields

$$\begin{aligned}E\left\{\hat{f}_-(c)\right\} &= f_-(c) + bf_-^{(1)}(c)\left\{\frac{\gamma(a+2,a)}{\gamma(a+1,a)} - a\right\} \\ &\quad + \frac{b^2}{2}f_-^{(2)}(c)\left\{\frac{\gamma(a+3,a)}{\gamma(a+1,a)} - 2a\frac{\gamma(a+2,a)}{\gamma(a+1,a)} + a^2\right\} + R_{\hat{f}_-(c)},\end{aligned} \tag{A7}$$

where

$$R_{\hat{f}_-(c)} := \frac{b^2}{2}\int_0^a \left\{f^{(2)}(\xi) - f_-^{(2)}(c)\right\}(v-a)^2\left\{\frac{v^a\exp(-v)}{\gamma(a+1,a)}\right\} dv$$

is the remainder term with $\xi = \theta(bv) + (1-\theta)c$ for some $\theta \in (0,1)$.

We approximate the leading bias terms first. Using (A2) recursively, we have

$$\begin{aligned}\gamma(a+2,a) &= (a+1)\gamma(a+1,a) - a^{a+1}\exp(-a), \text{ and} \\ \gamma(a+3,a) &= (a+2)(a+1)\gamma(a+1,a) - 2(a+1)a^{a+1}\exp(-a).\end{aligned}$$

It follows from (A1) and (A6) that

$$\begin{aligned}\frac{\gamma(a+2,a)}{\gamma(a+1,a)} - a &= 1 - \frac{a^{a+1}\exp(-a)}{\Gamma(a+1)}\left\{\frac{\gamma(a+1,a)}{\Gamma(a+1)}\right\}^{-1} \\ &= -\sqrt{\frac{2}{\pi}}a^{1/2} + \left(1 - \frac{4}{3\pi}\right) + O\left(a^{-1/2}\right), \text{ and} \\ \frac{\gamma(a+3,a)}{\gamma(a+1,a)} - 2a\frac{\gamma(a+2,a)}{\gamma(a+1,a)} + a^2 &= a + 2 - 2\frac{a^{a+1}\exp(-a)}{\Gamma(a+1)}\left\{\frac{\gamma(a+1,a)}{\Gamma(a+1)}\right\}^{-1} \\ &= a + O\left(a^{1/2}\right).\end{aligned}$$

Substituting these into the second and third terms on the right-hand side of (A7) and recognizing that $a = c/b$, we obtain

$$\begin{aligned}& bf_-^{(1)}(c)\left\{\frac{\gamma(a+2,a)}{\gamma(a+1,a)} - a\right\} + \frac{b^2}{2}f_-^{(2)}(c)\left\{\frac{\gamma(a+3,a)}{\gamma(a+1,a)} - 2a\frac{\gamma(a+2,a)}{\gamma(a+1,a)} + a^2\right\} \\ = \; & -\sqrt{\frac{2}{\pi}}c^{1/2}f_-^{(1)}(c)\,b^{1/2} + \left\{\left(1 - \frac{4}{3\pi}\right)f_-^{(1)}(c) + \frac{c}{2}f_-^{(2)}(c)\right\}b + o(b).\end{aligned}$$

The remaining task is to demonstrate that $R_{\hat{f}_-(c)} = o(b)$. It follows from Hölder-continuity of $f^{(2)}(\cdot)$ and $v \le c/b = a$ that

$$\left|f^{(2)}(\xi) - f_-^{(2)}(c)\right| \le L|\xi - c|^{\varsigma} = L\theta^{\varsigma}b^{\varsigma}(a-v)^{\varsigma}.$$

Using Hölder's inequality and the fact that $v^a\exp(-v)/\gamma(a+1,a)$ is a density on $[0,a]$, we have

$$\begin{aligned}\left|R_{\hat{f}_-(c)}\right| &\le \frac{L\theta^{\varsigma}}{2}b^{2+\varsigma}\int_0^a (a-v)^{2+\varsigma}\left\{\frac{v^a\exp(-v)}{\gamma(a+1,a)}\right\}dv \\ &\le \frac{L\theta^{\varsigma}}{2}b^{2+\varsigma}\left[\int_0^a (a-v)^3\left\{\frac{v^a\exp(-v)}{\gamma(a+1,a)}\right\}dv\right]^{(2+\varsigma)/3},\end{aligned}$$

where

$$\int_0^a (a-v)^3 \left\{ \frac{v^a \exp(-v)}{\gamma(a+1,a)} \right\} dv = a^3 - 3a^2 \frac{\gamma(a+2,a)}{\gamma(a+1,a)} + 3a \frac{\gamma(a+3,a)}{\gamma(a+1,a)} - \frac{\gamma(a+4,a)}{\gamma(a+1,a)}$$
$$= O\left(a^{3/2}\right)$$

by using (A1) and (A6) repeatedly. Finally, substituting $a = c/b$ yields

$$\left| R_{\hat{f}_-(c)} \right| \leq O\left(b^{2+\varsigma}\right) O\left\{ b^{-(1+\varsigma/2)} \right\} = O\left(b^{1+\varsigma/2}\right) = o(b),$$

which establishes the bias approximation.

**Variance.** In

$$Var\left\{ \hat{f}_-(c) \right\} = \frac{1}{n} E\left\{ K^-_{G(c,b;c)}(X_i) \right\}^2 + O\left(n^{-1}\right),$$

we make an approximation to $E\left\{ K^-_{G(c,b;c)}(X_i) \right\}^2$. By the change of variable $w := 2u/b$ and $a = c/b$,

$$E\left\{ K^-_{G(c,b;c)}(X_i) \right\}^2 = \int_0^c \frac{u^{2c/b} \exp(-2u/b)}{b^{2(c/b+1)} \gamma^2(c/b+1, c/b)} f(u)\, du$$
$$= b^{-1} \frac{\gamma(2a+1,2a)}{2^{2a+1} \gamma^2(a+1,a)} \int_0^{2a} f\left(\frac{bw}{2}\right) \left\{ \frac{w^{2a} \exp(-w)}{\gamma(2a+1,2a)} \right\} dw,$$

where the object inside brackets of the right-hand side is again a pdf. As before, the integral part can be approximated by $f_-(c) + O\left(b^{1/2}\right)$. Moreover, it follows from (A6), the argument on p.474 of Chen (2000) and $a = c/b$ that the multiplier part is

$$\left\{ \frac{\gamma(2a+1,2a)}{\Gamma(2a+1)} \right\} \left\{ \frac{\gamma(a+1,a)}{\Gamma(a+1)} \right\}^{-2} \left\{ \frac{b^{-1}\Gamma(2a+1)}{2^{2a+1}\Gamma^2(a+1)} \right\} = \frac{b^{-1/2}}{\sqrt{\pi} c^{1/2}} + o\left(b^{-1/2}\right).$$

Therefore,

$$Var\left\{ \hat{f}_-(c) \right\} = \frac{1}{nb^{1/2}} \frac{f_-(c)}{\sqrt{\pi} c^{1/2}} + o\left(n^{-1} b^{-1/2}\right). \blacksquare$$

## A.3 Proof of Theorem 1

The proof requires the following lemma.

***Lemma A1.***

$$E\left\{K^{\pm}_{G(c,b;c)}\left(X_i\right)\right\}^3 = O\left(b^{-1}\right).$$

### A.3.1 Proof of Lemma A1

To save space, we concentrate only on $E\left\{K^{-}_{G(c,b;c)}\left(X_i\right)\right\}^3$. By the change of variable $t := 3u/b$ and $a = c/b$,

$$\begin{aligned} E\left\{K^{-}_{G(c,b;c)}\left(X_i\right)\right\}^3 &= \int_0^c \frac{u^{3c/b}\exp\left(-3u/b\right)}{b^{3(c/b+1)}\gamma^3\left(c/b+1, c/b\right)} f\left(u\right)du \\ &= b^{-2}\frac{\gamma\left(3a+1,3a\right)}{3^{3a+1}\gamma^3\left(a+1,a\right)}\int_0^{3a} f\left(\frac{bt}{3}\right)\left\{\frac{t^{3a}\exp\left(-t\right)}{\gamma\left(3a+1,3a\right)}\right\}dt, \end{aligned}$$

where the integral part is $f_-\left(c\right) + O\left(b^{1/2}\right)$ as before. On the other hand, by (A1) and (A6), the multiplier part can be approximated by

$$\left\{\frac{\gamma\left(3a+1,3a\right)}{\Gamma\left(3a+1\right)}\right\}\left\{\frac{\gamma\left(a+1,a\right)}{\Gamma\left(a+1\right)}\right\}^{-3}\left\{\frac{b^{-2}\Gamma\left(3a+1\right)}{3^{3a+1}\Gamma^3\left(a+1\right)}\right\} = \frac{2}{\sqrt{3}\pi c}b^{-1} + o\left(b^{-1}\right),$$

which establishes the stated result. ■

### A.3.2 Proof of Theorem 1

Let

$$\begin{aligned} \hat{f}_{\pm,b}\left(c\right) &= E\left\{\hat{f}_{\pm,b}\left(c\right)\right\} + \left[\hat{f}_{\pm,b}\left(c\right) - E\left\{\hat{f}_{\pm,b}\left(c\right)\right\}\right] := I^{\pm}_b\left(c\right) + Z^{\pm}, \text{ and} \\ \hat{f}_{\pm,b/\delta}\left(c\right) &= E\left\{\hat{f}_{\pm,b/\delta}\left(c\right)\right\} + \left[\hat{f}_{\pm,b/\delta}\left(c\right) - E\left\{\hat{f}_{\pm,b/\delta}\left(c\right)\right\}\right] := I^{\pm}_{b/\delta}\left(c\right) + W^{\pm}. \end{aligned}$$

Then, by a similar argument to the proof for Theorem 1 of Hirukawa and Sakudo (2014) and Proposition 2,

$$\begin{aligned} \tilde{J}\left(c\right) &= \left\{I^{+}_b\left(c\right)\right\}^{\frac{1}{1-\delta^{1/2}}}\left\{I^{+}_{b/\delta}\left(c\right)\right\}^{-\frac{\delta^{1/2}}{1-\delta^{1/2}}} - \left\{I^{-}_b\left(c\right)\right\}^{\frac{1}{1-\delta^{1/2}}}\left\{I^{-}_{b/\delta}\left(c\right)\right\}^{-\frac{\delta^{1/2}}{1-\delta^{1/2}}} \\ &\quad + \left(\frac{1}{1-\delta^{1/2}}\right)\left\{\left(Z^{+} - \delta^{1/2}W^{+}\right) - \left(Z^{-} - \delta^{1/2}W^{-}\right)\right\} + R_{\tilde{J}(c)}, \end{aligned}$$

where the remainder term $R_{\tilde{J}(c)}$ is bounded by $\left|R_{\tilde{J}(c)}\right| \leq C\left(|Z^+| + |W^+| + |Z^-| + |W^-|\right)^2 = O_p\left(n^{-1}b^{-1/2}\right)$ for some constant $C \in (0, \infty)$. It follows from $E\left(Z^{\pm}\right) = E\left(W^{\pm}\right) = 0$ and Assumption 3 that

$$
\begin{aligned}
E\left\{\tilde{J}(c)\right\} &= \left\{I_b^+(c)\right\}^{\frac{1}{1-\delta^{1/2}}}\left\{I_{b/\delta}^+(c)\right\}^{-\frac{\delta^{1/2}}{1-\delta^{1/2}}} - \left\{I_b^-(c)\right\}^{\frac{1}{1-\delta^{1/2}}}\left\{I_{b/\delta}^-(c)\right\}^{-\frac{\delta^{1/2}}{1-\delta^{1/2}}} + O\left(n^{-1}b^{-1/2}\right) \\
&= J(c) + B(c)\,b + o(b),
\end{aligned}
$$

where

$$
\begin{aligned}
B(c) &= \left(\frac{1}{\delta^{1/2}}\right)\left[\frac{c}{\pi}\left\{\frac{\left(f_+^{(1)}(c)\right)^2}{f_+(c)} - \frac{\left(f_-^{(1)}(c)\right)^2}{f_-(c)}\right\}\right. \\
&\quad \left. -\left\{\left(1-\frac{4}{3\pi}\right)\left(f_+^{(1)}(c) - f_-^{(1)}(c)\right) + \frac{c}{2}\left(f_+^{(2)}(c) - f_-^{(2)}(c)\right)\right\}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sqrt{nb^{1/2}}\left\{\tilde{J}(c) - J(c)\right\} &= \sqrt{nb^{1/2}}\left[\tilde{J}(c) - E\left\{\tilde{J}(c)\right\}\right] + \sqrt{nb^{1/2}}\left[E\left\{\tilde{J}(c)\right\} - J(c)\right] \\
&= \sqrt{nb^{1/2}}\left(\frac{1}{1-\delta^{1/2}}\right)\left\{\left(Z^+ - \delta^{1/2}W^+\right) - \left(Z^- - \delta^{1/2}W^-\right)\right\} \\
&\quad + \sqrt{nb^{1/2}}\left\{B(c)\,b + o(b)\right\} + o_p(1),
\end{aligned}
$$

where the second term on the right hand side becomes asymptotically negligible if $nb^{5/2} \to 0$.

The remaining task is to establish the asymptotic normality of the first term. Due to the disjunction of two truncated kernels $K_{G(c,b;c)}^{\pm}(\cdot)$, the asymptotic variance of the term, denoted as $V(c)$, is just the sum of asymptotic variances of $\tilde{f}_{\pm}(c)$ given in Proposition 2. Hence, we need only to establish Liapunov's condition. Denoting

$$
\begin{aligned}
Z^{\pm} &= \sum_{i=1}^{n}\left(\frac{1}{n}\right)\left[K_{G(c,b;c)}^{\pm}(X_i) - E\left\{K_{G(c,b;c)}^{\pm}(X_i)\right\}\right] := \sum_{i=1}^{n}\left(\frac{1}{n}\right)Z_i^{\pm}, \text{ and} \\
W^{\pm} &= \sum_{i=1}^{n}\left(\frac{1}{n}\right)\left[K_{G(c,b/\delta;c)}^{\pm}(X_i) - E\left\{K_{G(c,b/\delta;c)}^{\pm}(X_i)\right\}\right] := \sum_{i=1}^{n}\left(\frac{1}{n}\right)W_i^{\pm},
\end{aligned}
$$

we can rewrite the term as

$$
\begin{aligned}
& \sqrt{nb^{1/2}}\left(\frac{1}{1-\delta^{1/2}}\right)\left\{\left(Z^{+}-\delta^{1/2}W^{+}\right)-\left(Z^{-}-\delta^{1/2}W^{-}\right)\right\} \\
= & \sum_{i=1}^{n}\sqrt{\frac{b^{1/2}}{n}}\left(\frac{1}{1-\delta^{1/2}}\right)\left\{\left(Z_i^{+}-\delta^{1/2}W_i^{+}\right)-\left(Z_i^{-}-\delta^{1/2}W_i^{-}\right)\right\}:=\sum_{i=1}^{n}Y_i.
\end{aligned}
$$

It follows from $0<\delta<1$ that

$$
E\left|Y_i\right|^3 \leq \frac{b^{3/4}}{n^{3/2}}\left(\frac{1}{1-\delta^{1/2}}\right)^3 E\left(\left|Z_i^{+}\right|+\left|W_i^{+}\right|+\left|Z_i^{-}\right|+\left|W_i^{-}\right|\right)^3.
$$

Because the expected value part is $O\left(b^{-1}\right)$ by Lemma A1, $E\left|Y_i\right|^3=O\left(n^{-3/2}b^{-1/4}\right)$. It is also straightforward to see that $Var\left(Y_i\right)=O\left(n^{-1}\right)$. Therefore,

$$
\frac{\sum_{i=1}^{n}E\left|Y_i\right|^3}{\left\{\sum_{i=1}^{n}Var\left(Y_i\right)\right\}^{3/2}}=O\left(n^{-1/2}b^{-1/4}\right)\to 0,
$$

or Liapunov's condition holds. This completes the proof. ■

## A.4 Proof of Proposition 3

The proof closely follows the one for Proposition 1 of Hirukawa and Sakudo (2016). It follows from Theorem 1 that $E\left\{\tilde{J}(c)\right\}=J(c)+O(b)$, $Var\left\{\tilde{J}(c)\right\}=O\left(n^{-1}b^{-1/2}\right)$ and $\tilde{V}(c)\xrightarrow{p}V(c)$, regardless of whether $H_0$ or $H_1$ may be true. Therefore, $\tilde{J}(c)=J(c)+O(b)+O_p\left(n^{-1/2}b^{-1/4}\right)\xrightarrow{p}J(c)\neq 0$ under $H_1$, and thus $|T(c)|$ is a divergent stochastic sequence with an expansion rate of $n^{1/2}b^{1/4}$. The result immediately follows. ■

## A.5 Proof of Theorem 2

To demonstrate this theorem, we must rely on different asymptotic expansions, depending on the positions of the design point $x$ and the truncation point $c$. For notational convenience, put $(a,z)=(x/b,c/b)$. The proof requires the following lemma.

***Lemma A2.*** *For* $a > 0$ *and* $z > \max\{1, a\}$,

$$\Gamma(a+1,z) \le \begin{cases} z^a \exp(-z) + \exp(-z) & \text{for } 0 < a \le 1 \\ (a+1) z^a \exp(-z) + \Gamma(a+1)\exp(-z) & \text{for } a > 1 \end{cases}.$$

### A.5.1 Proof of Lemma A2

For $0 < a \le 1$, it follows from an elementary inequality on the upper incomplete gamma function (e.g., equation (1.05) on p.67 of Olver, 1974) and $z > 1$ that

$$\Gamma(a,z) \le z^{a-1}\exp(-z) \le \exp(-z). \tag{A8}$$

Then, by (A3),

$$\Gamma(a+1,z) = z^a \exp(-z) + a\Gamma(a,z) \le z^a \exp(-z) + 1 \cdot \exp(-z).$$

Next, for $a > 1$ and $a \in \mathbb{N}$, using (A3) recursively yields

$$\begin{aligned} \Gamma(a+1,z) &= z^a \exp(-z)\left\{1 + \frac{a}{z} + \frac{a(a-1)}{z^2} + \cdots + \frac{a(a-1)\cdots 2}{z^{a-1}}\right\} \\ &\quad + a(a-1)\cdots 2 \cdot 1 \cdot \Gamma(1,z), \end{aligned}$$

where the sum inside the brackets is bounded by $a\,(\le a+1)$. Then, by (A8),

$$\Gamma(a+1,z) \le (a+1) z^a \exp(-z) + \Gamma(a+1)\exp(-z).$$

Finally, for $a > 1$ and $a \notin \mathbb{N}$, we have

$$\begin{aligned} \Gamma(a+1,z) &= z^a \exp(-z)\left\{1 + \frac{a}{z} + \frac{a(a-1)}{z^2} + \cdots + \frac{a(a-1)\cdots(a-\lfloor a\rfloor+1)}{z^{\lfloor a\rfloor}}\right\} \\ &\quad + a(a-1)\cdots(a-\lfloor a\rfloor)\Gamma(a-\lfloor a\rfloor, z). \end{aligned}$$

where the sum inside the brackets is bounded by $\lfloor a\rfloor + 1\,(\le a+1)$. Because $0 < a - \lfloor a\rfloor < 1$, $\Gamma(a-\lfloor a\rfloor) > 1$ and thus

$$a(a-1)\cdots(a-\lfloor a\rfloor) = \frac{\Gamma(a+1)}{\Gamma(a-\lfloor a\rfloor)} \le \Gamma(a+1).$$

Therefore, again by (A8),

$$\Gamma(a+1,z) \le (a+1) z^a \exp(-z) + \Gamma(a+1)\exp(-z). \quad \blacksquare$$

### A.5.2 Proof of Theorem 2

**(i) On $\hat{f}_{-}(x)$:**

We consider different approximations to incomplete gamma functions depending on the position of $x$. When $x/b \to \infty$, $z > a$ and $a, z \to \infty$ hold. Hence, the case for $a > 1$ of Lemma A2 applies, and thus

$$\frac{\Gamma(a+1,z)}{\Gamma(a+1)} \leq (a+1)\left\{\frac{z^a \exp(-z)}{\Gamma(a+1)}\right\} + \exp(-z).$$

It follows from (A1) and $\rho := a/z \in (0,1)$ that

$$\begin{aligned}\frac{z^a \exp(-z)}{\Gamma(a+1)} &= \left\{\frac{1+O(a^{-1})}{\sqrt{2\pi}}\right\} a^{-1/2} \exp\left\{a \ln\left(\frac{e}{\rho e^{1/\rho}}\right)\right\} \\ &= O\left[a^{-1/2} \exp\left\{a \ln\left(\frac{e}{\rho e^{1/\rho}}\right)\right\}\right], \end{aligned} \tag{A9}$$

where $e/\left(\rho e^{1/\rho}\right) \in (0,1)$ holds. Then,

$$\frac{\Gamma(a+1,z)}{\Gamma(a+1)} = O\left[a^{1/2} \exp\left\{a \ln\left(\frac{e}{\rho e^{1/\rho}}\right)\right\}\right].$$

On the other hand, when $x/b \to \kappa \in (0,\infty)$, putting $a \to \kappa$ and $z \to \infty$ in Lemma A2 yields

$$\frac{\Gamma(a+1,z)}{\Gamma(a+1)} = O\left\{z^{\kappa} \exp(-z)\right\}.$$

By (A4), we finally have

$$\frac{\gamma(a+1,z)}{\Gamma(a+1)} = 1 + \begin{cases} O\left[a^{1/2} \exp\left\{a \ln\left(e/\left(\rho e^{1/\rho}\right)\right)\right\}\right] & \text{if } x/b \to \infty \\ O\left\{z^{\kappa} \exp(-z)\right\} & \text{if } x/b \to \kappa \end{cases}. \tag{A10}$$

**Bias.** By (A9), (A10), and $(a,z) = (x/b, c/b)$,

$$\begin{aligned}
&\frac{\gamma(a+2,z)}{\gamma(a+1,z)} - a \\
&= 1 - \frac{z^{a+1}\exp(-z)}{\Gamma(a+1)}\left\{\frac{\gamma(a+1,z)}{\Gamma(a+1)}\right\}^{-1} \\
&= 1 + \begin{cases} O\left[a^{1/2}\exp\left\{a\ln\left(e/\left(\rho e^{1/\rho}\right)\right)\right\}\right] & \text{if } x/b \to \infty \\ O\left\{z^{\kappa}\exp(-z)\right\} & \text{if } x/b \to \kappa \end{cases} \\
&= 1 + \begin{cases} O\left[b^{-1/2}\exp\left\{(x/b)\ln\left(e/\left(\rho e^{1/\rho}\right)\right)\right\}\right] & \text{if } x/b \to \infty \\ O\left\{b^{-\kappa}\exp(-c/b)\right\} & \text{if } x/b \to \kappa \end{cases}, \text{ and} \\
&\frac{\gamma(a+3,z)}{\gamma(a+1,z)} - 2a\frac{\gamma(a+2,z)}{\gamma(a+1,z)} + a^2 \\
&= a + 2 - (z - a + 2)\frac{z^{a+1}\exp(-z)}{\Gamma(a+1)}\left\{\frac{\gamma(a+1,z)}{\Gamma(a+1)}\right\}^{-1} \\
&= a + 2 + \begin{cases} O\left[a^{3/2}\exp\left\{a\ln\left(e/\left(\rho e^{1/\rho}\right)\right)\right\}\right] & \text{if } x/b \to \infty \\ O\left\{z^{\kappa+1}\exp(-z)\right\} & \text{if } x/b \to \kappa \end{cases} \\
&= \frac{x}{b} + 2 + \begin{cases} O\left[b^{-3/2}\exp\left\{(x/b)\ln\left(e/\left(\rho e^{1/\rho}\right)\right)\right\}\right] & \text{if } x/b \to \infty \\ O\left\{b^{-\kappa-1}\exp(-c/b)\right\} & \text{if } x/b \to \kappa \end{cases}.
\end{aligned}$$

Then, by the argument in the proof of Proposition 1, in either case,

$$E\left\{\hat{f}_-(x)\right\} = f(x) + \left\{f^{(1)}(x) + \frac{x}{2}f^{(2)}(x)\right\}b + o(b).$$

**Variance.** In

$$E\left\{K^{-}_{G(x,b;c)}(X_i)\right\}^2 = b^{-1}\frac{\gamma(2a+1,2z)}{2^{2a+1}\gamma^2(a+1,z)}\int_0^{2z} f\left(\frac{bw}{2}\right)\left\{\frac{w^{2a}\exp(-w)}{\gamma(2a+1,2z)}\right\}dw,$$

the integral part is $f(x) + O(b)$ in either case. It also follows from (A10) and the argument on p.474 of Chen (2000) that the multiplier part is

$$\begin{aligned}
&\left\{\frac{\gamma(2a+1,2z)}{\Gamma(2a+1)}\right\}\left\{\frac{\gamma(a+1,z)}{\Gamma(a+1)}\right\}^{-2}\left\{\frac{b^{-1}\Gamma(2a+1)}{2^{2a+1}\Gamma^2(a+1)}\right\} \\
&= \begin{cases} \frac{b^{-1/2}}{2\sqrt{\pi}x^{1/2}} + o\left(b^{-1/2}\right) & \text{if } x/b \to \infty \\ \frac{b^{-1}\Gamma(2\kappa+1)}{2^{2\kappa+1}\Gamma^2(\kappa+1)} + o\left(b^{-1}\right) & \text{if } x/b \to \kappa \end{cases}.
\end{aligned}$$

Therefore,

$$Var\left\{\hat{f}_-(x)\right\} = \begin{cases} \frac{1}{nb^{1/2}}\frac{f(x)}{2\sqrt{\pi}x^{1/2}} + o\left(n^{-1}b^{-1/2}\right) & \text{if } x/b \to \infty \\ \frac{1}{nb}\frac{\Gamma(2\kappa+1)}{2^{2\kappa+1}\Gamma^2(\kappa+1)}f(x) + o\left(n^{-1}b^{-1}\right) & \text{if } x/b \to \kappa \end{cases}. \blacksquare$$

**(ii) On $\hat{f}_{+}(x)$:**

We may focus only on the case for interior $x$. However, it seems difficult to derive a sharp bound on $\gamma(a+1,z)$ or $\Gamma(a+1,z)$ for the case of $a > z$ and $a, z \to \infty$ based directly on (A2) or (A3). Instead, we turn to the series expansion described in Section 3 of Ferreira, López and Pérez-Sinusía (2005), which is valid for the case of $a > z$, $a, z \to \infty$ and $a - z = O(a)$. The expansion is

$$\gamma(a+1,z) = z^{a+1}\exp(-z)\sum_{k=0}^{\infty} c_k(a)\,\Phi_k(z-a),$$

where the definitions of $\{c_k(a)\}$ and $\{\Phi_k(z-a)\}$ can be found therein. Because the sum is shown to be convergent, the order of magnitude in $\gamma(a+1,z)/\Gamma(a+1)$ is determined by the one in $z^{a+1}\exp(-z)/\Gamma(a+1)$. It follows from (A1) and $\rho' := z/a \in (0,1)$ that

$$\begin{aligned}\frac{z^{a+1}\exp(-z)}{\Gamma(a+1)} &= \left[\frac{\rho'\{1+O(a^{-1})\}}{\sqrt{2\pi}}\right] a^{1/2}\exp\left\{a\ln\left(\frac{\rho' e}{e^{\rho'}}\right)\right\} \\ &= O\left[a^{1/2}\exp\left\{a\ln\left(\frac{\rho' e}{e^{\rho'}}\right)\right\}\right],\end{aligned}$$

where $\rho' e/e^{\rho'} \in (0,1)$ is again the case. Then, by (A4),

$$\frac{\Gamma(a+1,z)}{\Gamma(a+1)} = 1 + O\left[a^{1/2}\exp\left\{a\ln\left(\frac{\rho' e}{e^{\rho'}}\right)\right\}\right].$$

The bias and variance of $\hat{f}_{+}(x)$ can be approximated as stated. ■

## A.6 Proof of Theorem 3

Both this proof and the proof of Theorem 4 require three lemmata below.

***Lemma A3.*** *For $\alpha > 0$ and a sufficiently small $b > 0$, pick some design point $x \in [0, \alpha b]$. Then, for $\eta \in (0, c)$,*

$$\int_0^{\eta} K_{G(x,b;c)}^{-}(u)\,du = \int_0^{\eta} \frac{u^{x/b}\exp(-u/b)}{b^{x/b+1}\gamma(x/b+1, c/b)}du \to 1$$

*as $b \to 0$.*

***Lemma A4.*** *For the design point $x$ defined in Lemma A3, let*

$$\{K_i\}_{i=1}^n := \left\{bK_{G(x,b;c)}^{-}(X_i)\right\}_{i=1}^n.$$

*Then,*

$$0 \le K_i \le C := \max\{1, \alpha^\alpha\}\left\{\frac{\Gamma(\alpha+1)}{\gamma(\alpha+1,\alpha)}\right\}\left\{\frac{1}{\Gamma(a^*)}\right\},$$

*where* $\Gamma(a^*) := \min_{a>0}\Gamma(a) \approx 0.8856$ *for* $a^* \approx 1.4616$.

***Lemma A5.(Hoeffding, 1963, Theorem 2)*** *Let $\{X_i\}_{i=1}^n$ be independent and $a_i \le X_i \le b_i$ for $i = 1, 2, \ldots, n$. Also write $\bar{X} := (1/n)\sum_{i=1}^n X_i$ and $\mu := E(\bar{X})$. Then, for $\epsilon > 0$,*

$$\Pr\left(\left|\bar{X}-\mu\right| \ge \epsilon\right) \le 2\exp\left\{-\frac{2n^2\epsilon^2}{\sum_{i=1}^n (b_i-a_i)^2}\right\}.$$

### A.6.1 Proof of Lemma A3

By the change of variable $v := u/b$, the integral can be rewritten as

$$\int_0^{\eta/b}\frac{v^{x/b}\exp(-v)}{\gamma(x/b+1,c/b)}dv = \frac{\gamma(x/b+1,\eta/b)}{\gamma(x/b+1,c/b)}.$$

Because $\eta/b \uparrow \infty$ and $0 \le x/b \le \alpha$, (A10) establishes that

$$\frac{\gamma(x/b+1,\eta/b)}{\gamma(x/b+1,c/b)} = \frac{\Gamma(x/b+1)+O\{b^{-\alpha}\exp(-\eta/b)\}}{\Gamma(x/b+1)+O\{b^{-\alpha}\exp(-c/b)\}} \to 1. \blacksquare$$

### A.6.2 Proof of Lemma A4

By construction, $K_i \ge 0$ holds. In addition, since the gamma kernel has its mode at the design point $x$ (Chen, 2000, p.473), $K_i$ is bounded by

$$bK_{G(x,b;c)}^{-}(x) = \left(\frac{x}{b}\right)^{x/b}\exp\left(-\frac{x}{b}\right)\left\{\frac{\Gamma(x/b+1)}{\gamma(x/b+1,c/b)}\right\}\left\{\frac{1}{\Gamma(x/b+1)}\right\}. \quad \text{(A11)}$$

For $0 \le x/b \le \alpha$, $(x/b)^{x/b} \le \max\{1,\alpha^\alpha\}$ and $\exp(-x/b) \le 1$. Moreover, $\gamma(a,z)/\Gamma(a)$ for $a, z > 0$ is monotonously increasing in $z$ and decreasing in $a$; see, for example, Tricomi (1950, p.276) for details. Because $c$ is an interior point, $\alpha b \leq c$ or $\alpha \leq c/b$ holds. Hence,

$$\frac{\Gamma(x/b+1)}{\gamma(x/b+1, c/b)} \leq \frac{\Gamma(\alpha+1)}{\gamma(\alpha+1, \alpha)}.$$

Finally, it is known that $\Gamma(a^*) := \min_{a>0} \Gamma(a) \approx 0.8856$ for $a^* \approx 1.4616$. Therefore, the right-hand side of (A11) has the upper bound

$$\max\{1, \alpha^\alpha\} \cdot 1 \cdot \left\{\frac{\Gamma(\alpha+1)}{\gamma(\alpha+1, \alpha)}\right\} \left\{\frac{1}{\Gamma(a^*)}\right\} := C. \blacksquare$$

### A.6.3 Proof of Theorem 3

This proof largely follows the one for Theorem 5 of Hirukawa and Sakudo (2015). Without loss of generality, for $\alpha > 0$ and a sufficiently small $b > 0$, pick some design point $x \in [0, \alpha b]$. Then, the proof completes if the following statements hold:

$$\hat{f}_-(x) = E\left\{\hat{f}_-(x)\right\} + o_p(1). \tag{A12}$$

$$E\left\{\hat{f}_-(x)\right\} = E\left\{\hat{f}_-(0)\right\} + o(1). \tag{A13}$$

$$E\left\{\hat{f}_-(0)\right\} \to \infty. \tag{A14}$$

Below we demonstrate (A12)-(A14) one by one. First, (A13) immediately follows from the continuity of $K^-_{G(x,b;c)}(u)$ in $x$. Second, when $f(x) \to \infty$ as $x \to 0$, it holds that for any $A > 0$, there is some $\eta \in (0, c)$ such that $f(x) > A$ for all $x < \eta$. For the given $\eta$, Lemma A3 implies that

$$E\left\{\hat{f}_-(0)\right\} > \int_0^\eta K^-_{G(0,b;c)}(u) f(u)\, du > A \int_0^\eta K^-_{G(0,b;c)}(u)\, du \to A,$$

which establishes (A14). Third, for $\{K_i\}_{i=1}^n$ defined in Lemma A4, denote their sample average as $\bar{K} := (1/n) \sum_{i=1}^n K_i$. Then, it follows from Lemmata A4 and A5 that for $\epsilon > 0$,

$$\begin{aligned} \Pr\left(\left|\hat{f}_-(x) - E\left\{\hat{f}_-(x)\right\}\right| \geq \epsilon\right) &= \Pr\left(\left|\bar{K} - E(K_i)\right| \geq b\epsilon\right) \\ &\leq 2\exp\left\{-2\left(\frac{\epsilon}{C}\right)^2 nb^2\right\} \to 0. \end{aligned}$$

Therefore, (A12) is also demonstrated, and thus the proof is completed. ■

## A.7 Proof of Theorem 4

This proof largely follows the one for Theorem 5.3 of Bouezmarni and Scaillet (2005). As in the proof of Theorem 3, pick some $x \in [0, \alpha b]$. Then, the proof is boiled down to establishing the following statements:

$$\left| \frac{E\left\{\hat{f}_{-}(x)\right\} - f(x)}{f(x)} \right| \to 0, \text{ and} \tag{A15}$$

$$\left| \frac{\hat{f}_{-}(x) - E\left\{\hat{f}_{-}(x)\right\}}{f(x)} \right| \xrightarrow{p} 0, \tag{A16}$$

as $n \to \infty$ and $b, x \to 0$.

We demonstrate (A15) first. An inspection of the proof for Theorem 5.3 of Bouezmarni and Scaillet (2005) reveals that (A15) is shown if their conditions A.2, A.3 and A.5 are fulfilled. Now we check the validity of three conditions. First, because $\int_0^\infty f(x)\,dx = 1$ and $f(x) \to \infty$ as $x \to 0$, there are constants $0 < \underline{C} < \overline{C} < \infty$ such that $\underline{C}x^{-d} \le f(x) \le \overline{C}x^{-d}$ for some $d \in (0,1)$ as $x \to 0$. Accordingly, $f^{(1)}(x) = O\left(x^{-d-1}\right)$ for a small value of $x$. These imply that $x\left|f^{(1)}(x)\right|/f(x) \le O(1)$, and thus A.2 follows. Second, A.3 has been already established as Lemma A1. Third, let the random variable $U$ be drawn from the distribution with the pdf $K^{-}_{G(x,b;c)}(u)$. Then, by $0 \le x/b \le \alpha$ and the expansion techniques used in the proof of Theorem 2, $Var(U) \le O(b) \to 0$, and thus A.5 also holds.

Furthermore, it follows from Lemmata A4 and A5 that for $\bar{K}$ defined in the proof of Theorem 3 and for $\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\left| \frac{\hat{f}_{-}(x) - E\left\{\hat{f}_{-}(x)\right\}}{f(x)} \right| \ge \epsilon\right) &= \Pr\left(\left|\bar{K} - E(K_i)\right| \ge bf(x)\epsilon\right) \\
&\le 2\exp\left\{-2\left(\frac{\epsilon}{C}\right)^2 nb^2 f^2(x)\right\} \to 0.
\end{aligned}$$

Therefore, (A16) is also demonstrated, and thus the proof is completed. ■

# B Additional simulation results

This section presents results from an additional Monte Carlo study. Some readers may wonder how sensitive finite-sample properties of our proposed test statistics $T_1(c)$ and $T_2(c)$ are to the choices of two exponents $(p, q)$ in the power-optimality smoothing parameter selection method. Then, we replicate the simulation study in Section 4.2 by changing only one of the values of $(p, q)$. The benchmark case is $(p, q) = (1/2, 4/9)$, and $1/6$ is either added to or subtracted from each benchmark value to conduct sensitivity analyses. For $p$, $1/2 \pm 1/6 = 1/3, 2/3$ are considered. For $q$, because $4/9$ is close to the lower bound of $(2/5, 2/3)$ (= the admissible range for $q$), only $4/9 + 1/6 = 11/18$ is examined. The mixing exponent $\delta = 0.81$ is maintained, and 5000 replications of Monte Carlo samples with the sample size $n = 1000$ are drawn. All other details in the Monte Carlo design follow those given in Section 4.2.

Tables B1 and B2 report the results with various $p$ and $q$, respectively. For each table, the results for the benchmark case are the same as in Table 3. Table B1 indicates that cutting down $p$ (or adopting a small number of sub-samples) considerably ameliorates power properties of the tests. However, such power improvement is often accompanied with severe size distortions; see the results for $(p, d) = (1/3, 0.00)$ in the cases of choosing the 30% quantile as the cutoff. From the viewpoint of the balance between size and power properties, $p = 1/2$ looks reasonable. In addition, a larger $q$ (i.e., employing a smaller smoothing parameter value or undersmoothing) is expected to yield a wider confidence interval, which in turn leads to power loss. Table B2 ensures this aspect numerically, indicating that $q = 4/9$ is better. In sum, two tables jointly suggest that $(p, q) = (1/2, 4/9)$ are indeed safe choices.

**Table B1:** Finite-sample power properties of test statistics for discontinuity with various $p$ $[n = 1000; \delta = 0.81; q = 4/9]$

(%)

| | | | | | $d$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Distribution | $c$ | Test | $p$ | Nominal | 0.00 | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 |
| Gamma | 1.7057 | $T_1(c)$ | 1/3 | 5% | 4.4 | 37.8 | 86.1 | 99.2 | 99.9 | 99.9 |
| | (30%) | | | 10% | 9.0 | 40.8 | 87.9 | 99.5 | 100.0 | 100.0 |
| | | | 1/2 | 5% | 3.9 | 6.6 | 12.8 | 37.1 | 98.7 | 100.0 |
| | | | | 10% | 8.2 | 12.4 | 21.4 | 46.0 | 99.0 | 100.0 |
| | | | 2/3 | 5% | 3.9 | 6.7 | 12.8 | 26.0 | 45.4 | 98.6 |
| | | | | 10% | 8.3 | 12.4 | 21.7 | 38.2 | 58.4 | 99.0 |
| | | $T_2(c)$ | 1/3 | 5% | 20.6 | 67.9 | 91.5 | 98.4 | 99.4 | 99.8 |
| | | | | 10% | 25.8 | 79.2 | 96.8 | 99.8 | 99.9 | 100.0 |
| | | | 1/2 | 5% | 4.4 | 13.9 | 50.3 | 90.8 | 99.5 | 100.0 |
| | | | | 10% | 8.9 | 19.2 | 54.9 | 92.7 | 99.9 | 100.0 |
| | | | 2/3 | 5% | 4.2 | 7.1 | 16.0 | 37.6 | 78.5 | 99.6 |
| | | | | 10% | 8.8 | 13.0 | 24.4 | 46.7 | 82.3 | 99.9 |
| | 2.4248 | $T_1(c)$ | 1/3 | 5% | 3.7 | 5.4 | 12.3 | 26.7 | 44.0 | 69.3 |
| | (Med) | | | 10% | 7.7 | 10.8 | 21.3 | 38.5 | 58.1 | 78.6 |
| | | | 1/2 | 5% | 3.9 | 5.0 | 10.4 | 20.7 | 35.5 | 53.2 |
| | | | | 10% | 8.0 | 10.4 | 18.3 | 32.1 | 49.0 | 65.8 |
| | | | 2/3 | 5% | 4.0 | 5.1 | 9.5 | 18.2 | 31.0 | 47.7 |
| | | | | 10% | 8.2 | 10.1 | 17.2 | 29.0 | 44.3 | 61.2 |
| | | $T_2(c)$ | 1/3 | 5% | 4.3 | 6.2 | 13.7 | 32.4 | 58.7 | 91.4 |
| | | | | 10% | 8.6 | 11.8 | 22.7 | 42.5 | 67.7 | 93.0 |
| | | | 1/2 | 5% | 4.3 | 5.6 | 11.3 | 22.2 | 36.8 | 55.0 |
| | | | | 10% | 8.6 | 11.1 | 19.2 | 33.1 | 50.3 | 67.0 |
| | | | 2/3 | 5% | 4.3 | 5.4 | 10.3 | 19.2 | 32.2 | 48.9 |
| | | | | 10% | 8.7 | 10.8 | 17.8 | 29.9 | 45.0 | 62.0 |
| Weibull | 1.9419 | $T_1(c)$ | 1/3 | 5% | 8.8 | 53.3 | 88.9 | 98.9 | 99.6 | 99.8 |
| | (30%) | | | 10% | 12.7 | 57.6 | 92.1 | 99.6 | 100.0 | 100.0 |
| | | | 1/2 | 5% | 4.2 | 6.5 | 12.9 | 42.1 | 98.4 | 99.9 |
| | | | | 10% | 8.4 | 12.4 | 21.2 | 49.1 | 99.1 | 100.0 |
| | | | 2/3 | 5% | 4.2 | 6.3 | 11.1 | 23.0 | 40.7 | 97.6 |
| | | | | 10% | 8.3 | 12.3 | 20.0 | 34.8 | 53.1 | 98.3 |
| | | $T_2(c)$ | 1/3 | 5% | 29.8 | 65.5 | 87.7 | 96.7 | 98.7 | 99.6 |
| | | | | 10% | 38.0 | 80.1 | 95.1 | 99.3 | 99.8 | 100.0 |
| | | | 1/2 | 5% | 5.2 | 17.5 | 51.0 | 88.5 | 98.9 | 99.9 |
| | | | | 10% | 9.4 | 23.2 | 56.5 | 91.1 | 99.7 | 100.0 |
| | | | 2/3 | 5% | 4.5 | 6.6 | 14.4 | 30.7 | 65.9 | 98.9 |
| | | | | 10% | 8.8 | 12.9 | 22.4 | 40.5 | 71.5 | 99.6 |
| | 2.8386 | $T_1(c)$ | 1/3 | 5% | 3.9 | 5.7 | 12.9 | 33.1 | 60.6 | 92.1 |
| | (Med) | | | 10% | 7.9 | 11.4 | 22.0 | 42.7 | 68.6 | 93.5 |
| | | | 1/2 | 5% | 3.8 | 5.1 | 10.2 | 20.2 | 34.3 | 50.9 |
| | | | | 10% | 8.2 | 10.7 | 18.1 | 31.5 | 47.3 | 63.8 |
| | | | 2/3 | 5% | 3.9 | 5.3 | 9.5 | 17.8 | 29.6 | 45.7 |
| | | | | 10% | 8.3 | 10.1 | 17.0 | 28.0 | 42.9 | 59.0 |
| | | $T_2(c)$ | 1/3 | 5% | 4.3 | 7.3 | 25.2 | 70.8 | 91.5 | 99.7 |
| | | | | 10% | 8.7 | 12.7 | 31.8 | 73.4 | 92.7 | 99.8 |
| | | | 1/2 | 5% | 4.2 | 5.7 | 11.3 | 21.3 | 36.7 | 61.7 |
| | | | | 10% | 8.6 | 11.3 | 19.0 | 32.5 | 48.9 | 70.6 |
| | | | 2/3 | 5% | 4.3 | 5.6 | 9.9 | 18.5 | 30.5 | 46.6 |
| | | | | 10% | 8.8 | 10.7 | 17.6 | 28.8 | 43.6 | 59.7 |

**Table B2:** Finite-sample power properties of test statistics for discontinuity with various $q$ $[n = 1000; \delta = 0.81; p = 1/2]$

(%)

| | | | | | d | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Distribution | $c$ | Test | $q$ | Nominal | 0.00 | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 |
| Gamma | 1.7057 (30%) | $T_1(c)$ | 4/9 | 5% | 3.9 | 6.6 | 12.8 | 37.1 | 98.7 | 100.0 |
| | | | | 10% | 8.2 | 12.4 | 21.4 | 46.0 | 99.0 | 100.0 |
| | | | 11/18 | 5% | 4.3 | 6.3 | 11.2 | 35.2 | 93.6 | 99.2 |
| | | | | 10% | 8.3 | 12.1 | 20.0 | 43.6 | 96.7 | 99.8 |
| | | $T_2(c)$ | 4/9 | 5% | 4.4 | 13.9 | 50.3 | 90.8 | 99.5 | 100.0 |
| | | | | 10% | 8.9 | 19.2 | 54.9 | 92.7 | 99.9 | 100.0 |
| | | | 11/18 | 5% | 4.7 | 11.3 | 37.6 | 76.8 | 92.5 | 98.7 |
| | | | | 10% | 9.0 | 18.1 | 47.4 | 85.1 | 96.7 | 99.6 |
| | 2.4248 (Med) | $T_1(c)$ | 4/9 | 5% | 3.9 | 5.0 | 10.4 | 20.7 | 35.5 | 53.2 |
| | | | | 10% | 8.0 | 10.4 | 18.3 | 32.1 | 49.0 | 65.8 |
| | | | 11/18 | 5% | 4.1 | 5.1 | 9.4 | 18.3 | 30.7 | 46.7 |
| | | | | 10% | 8.2 | 10.0 | 17.1 | 29.0 | 44.1 | 60.3 |
| | | $T_2(c)$ | 4/9 | 5% | 4.3 | 5.6 | 11.3 | 22.2 | 36.8 | 55.0 |
| | | | | 10% | 8.6 | 11.1 | 19.2 | 33.1 | 50.3 | 67.0 |
| | | | 11/18 | 5% | 4.3 | 5.4 | 10.1 | 19.3 | 31.9 | 48.2 |
| | | | | 10% | 8.8 | 10.6 | 17.8 | 30.1 | 44.6 | 61.5 |
| Weibull | 1.9419 (30%) | $T_1(c)$ | 4/9 | 5% | 4.2 | 6.5 | 12.9 | 42.1 | 98.4 | 99.9 |
| | | | | 10% | 8.4 | 12.4 | 21.2 | 49.1 | 99.1 | 100.0 |
| | | | 11/18 | 5% | 4.4 | 6.2 | 12.4 | 39.7 | 90.9 | 98.5 |
| | | | | 10% | 8.4 | 11.9 | 19.7 | 47.1 | 95.1 | 99.4 |
| | | $T_2(c)$ | 4/9 | 5% | 5.2 | 17.5 | 51.0 | 88.5 | 98.9 | 99.9 |
| | | | | 10% | 9.4 | 23.2 | 56.5 | 91.1 | 99.7 | 100.0 |
| | | | 11/18 | 5% | 5.1 | 12.9 | 35.8 | 70.8 | 88.2 | 97.1 |
| | | | | 10% | 9.2 | 19.3 | 46.1 | 80.7 | 94.2 | 99.1 |
| | 2.8386 (Med) | $T_1(c)$ | 4/9 | 5% | 3.8 | 5.1 | 10.2 | 20.2 | 34.3 | 50.9 |
| | | | | 10% | 8.2 | 10.7 | 18.1 | 31.5 | 47.3 | 63.8 |
| | | | 11/18 | 5% | 3.8 | 5.3 | 9.3 | 17.7 | 29.3 | 44.7 |
| | | | | 10% | 8.5 | 10.1 | 16.8 | 28.1 | 42.5 | 57.9 |
| | | $T_2(c)$ | 4/9 | 5% | 4.2 | 5.7 | 11.3 | 21.3 | 36.7 | 61.7 |
| | | | | 10% | 8.6 | 11.3 | 19.0 | 32.5 | 48.9 | 70.6 |
| | | | 11/18 | 5% | 4.4 | 5.6 | 9.9 | 18.4 | 31.6 | 57.8 |
| | | | | 10% | 8.8 | 10.7 | 17.5 | 28.8 | 43.9 | 66.8 |

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