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On complete consistency for the weighted estimator of nonparametric regression models. (English summary)

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This article aims at demonstrating complete convergence [P. L. Hsü and H. E. Robbins, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25–31; MR0019852] of a class of nonparametric regression estimators in which regression errors obey a newly proposed dependence structure. The regression model of interest is of the form

$$Y_{ni} = f(x_{ni}) + \varepsilon_{ni}, \quad i = 1, \dots, n,$$

where x_{ni} are known fixed design points in a compact set $A \subset \mathbb{R}^d$, $f(\cdot)$ is an unknown regression function, and ε_{ni} are widely orthant dependent (WOD) [K. Wang, Y. B. Wang and Q. Gao, Methodol. Comput. Appl. Probab. **15** (2013), no. 1, 109–124; MR3030214] random variables with mean zero. The weighted regression estimator

$$f_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, \quad x \in A,$$

is exclusively considered, where $W_{ni}(x) = W_{ni}(x; x_{n1}, \dots, x_{nn})$ are weight functions.

Before proceeding, definitions of complete convergence and WOD should be provided. First, a random sequence Y_n, $n = 1, 2, \dots$ is said to converge completely to a constant Y if

$$\sum_{n=1}^{\infty} \Pr(|Y_n - Y| > \epsilon) < \infty$$

holds for every $\epsilon > 0$. Complete convergence is stronger than almost sure convergence in that $Y_n \to Y$ *completely* as $n \to \infty$ establishes $Y_n \to Y$ *a.s.* by the Borel-Cantelli lemma. Second, WOD is built on two dependence structures. Random variables $X_1, \dots, X_n$ are said to be widely upper orthant dependent (WUOD) if there is a finite real number $g_U(n)$ so that

$$\Pr\left\{\bigcap_{i=1}^{n} (X_i > x_i)\right\} \le g_U(n) \prod_{i=1}^{n} \Pr(X_i > x_i)$$

holds for every $x_i \in (-\infty, \infty)$, $i = 1, \dots, n$. In addition, $X_1, \dots, X_n$ are said to be widely lower orthant dependent (WLOD) if there is a finite real number $g_L(n)$ so that

$$\Pr\left\{\bigcap_{i=1}^{n} (X_i \le x_i)\right\} \le g_L(n) \prod_{i=1}^{n} \Pr(X_i \le x_i)$$

holds for every $x_i \in (-\infty, \infty)$, $i = 1, \dots, n$. Finally, $X_1, \dots, X_n$ are said to be WOD if they are both WUOD and WLOD. These dependence structures were proposed originally in the context of a stochastic risk model that can describe ruin probabilities of insurance companies.

The main result of the article under review is that $f_n(x) \to f(x)$ *completely* as $n \to$

∞ for every continuity point x of the function f on A if all the following regularity conditions are satisfied:

(1) $\sum_{i=1}^{n} W_{ni}(x) \to 1$ as $n \to \infty$.
(2) $\sum_{i=1}^{n} |W_{ni}(x)| \le C < \infty$ for every n.
(3) $\sum_{i=1}^{n} |W_{ni}(x)||f(x_{ni}) - f(x)|\mathbf{1}(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for every $a > 0$.
(4) For some $p \ge 1$,
 (a) $\max_{1 \le i \le n} |W_{ni}(x)| = O(n^{-1/p})$; and
 (b) $E|X|^{2p} < \infty$ for a random variable X that stochastically dominates ε_{ni}.
(5) $g(n) = \max\{g_U(n), g_L(n)\} = O(n^{\lambda})$ for some $\lambda \ge 0$.

The result is an improvement over Theorem 4.1 of X. J. Wang et al. [TEST **23** (2014), no. 3, 607–629; MR3252097]. The theorem establishes complete convergence of $f_n(x)$ under stronger assumptions. More specifically, stronger conditions are imposed in place of (4)(b) and (5) above, whereas all other conditions remain unchanged.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.