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Consistency of a nonparametric least squares estimator in integer-valued GARCH models. (English summary)
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In this article, the convergence rate of a least squares estimator of a nonparametric integer-valued GARCH (INGARCH) model is explored. An INGARCH process can be represented as a bivariate one $\{(\lambda_t, Y_t)\}$, where a time series of count variables $\{Y_t\}$ obeys a conditional Poisson distribution and the intensity $\{\lambda_t\}$ is modeled as a recursive relation.

A nonparametric version of INGARCH processes is studied in this article. Let $\mathcal{F}_t := \sigma\{\lambda_s, Y_s : s \le t\}$ be the σ-field generated by the process up to time t. Also, let m be an unknown, nonnegative link function. Then, the INGARCH process concerned is modeled as

$$Y_t|\,\mathcal{F}_{t-1} \sim Poiss\,(\lambda_t) \text{ and } \lambda_t = m\,(Y_{t-1}, \dots, Y_{t-p}, \lambda_{t-1}, \dots, \lambda_{t-q})\,.$$

In this model, the process $\{Y_t\}$ obeys a nonparametric INGARCH(p, q) count process with accompanying intensity process $\{\lambda_t\}$.

A particular focus is on an INGARCH$(1, 1)$ process with the intensity $\lambda_t = m(\lambda_{t-1}, Y_{t-1})$. A least squares estimation of the unknown function m is investigated. Existence of a stationarity distribution π of the bivariate process $\{(\lambda_t, Y_t)\}$ is ensured if m is bounded in λ and smooth (or contractive, to be more precise) in both arguments. To approximate m, consider a class of function $g^{[i]}$ that satisfies

$$g^{[0]}\,(\lambda, Y_t) = g\,(\lambda, Y_t) \text{ and } g^{[k]}\,(\lambda, Y_t, \dots, Y_{t+k}) = g\left\{g^{[k-1]}\,(\lambda, Y_t, \dots, Y_{t+k-1})\,, Y_{t+k}\right\}.$$

Then, for given $(n+1)$ consecutive observations $\{Y_t\}_{t=0}^{n}$, the least squares estimator of m, denoted by $\widehat{m}_n$, can be obtained as $g^{[i]}$ that minimizes the empirical contrast functional $(1/n)\sum_{i=0}^{n-1}\{Y_{i+1} - g^{[i]}(0, Y_0, \dots, Y_i)\}^2$. Consistency of $\widehat{m}_n$ for m is established by finding the convergence rate of its $L_2(\pi)$ loss

$$L\,(\widehat{m}_n, m) := \int \{\widehat{m}_n\,(\lambda, y) - m\,(\lambda, y)\}^2\,\pi\,(d\lambda, dy) = O_p\left(n^{-2/3}\log^2 n\right).$$

It is conjectured that the optimal rate of the $L_2(\pi)$ loss would be of order $n^{-2/3}$. The above convergence rate appears to be nearly optimal in light of this conjecture.

The convergence result documents a theoretical property of $\widehat{m}_n$ and does not provide its specific form. Sieve methods are recommended for the practical implementation of $\widehat{m}_n$. The article also refers to the possibility of estimating the link function m for general INGARCH(p, q) processes. *Masayuki Hirukawa*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.