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MR4830540 62G10 62C20 62H17

Kim, Ilmun (KR-YNS-SDS)**; Neykov, Matey** (1-NW-SDS)**;**
Balakrishnan, Sivaraman (1-CMU-SDS)**; Wasserman, Larry** (1-CMU-SDS)

Conditional independence testing for discrete distributions: beyond χ^2- and G-tests. (English. English summary)

Electron. J. Stat. **18** (2024), *no.* 2, 4767–4794.

This article studies the problem of testing conditional independence (CI) between two discrete random variables X and Y after accounting for another discrete random variable Z, given n i.i.d. observations $\{(X_i, Y_i, Z_i)\}_{i=1}^n$. The domain of Z is denoted as $[d] = \{1, \dots, d\}$, and throughout it is assumed that d is large relative to n. Accordingly, the CI testing procedure is constructed within a non-asymptotic framework.

The proposed CI test improves the one by C. L. Canonne et al. [in *STOC'18—Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, 735–748, ACM, New York, 2018; MR3826290]. Although the latter is also built on a finite-sample perspective, it has two difficulties in practice. First, it relies on Poissonization, where the sample size of the data drawn from (X, Y, Z) is a Poisson random variable. It is known that the testing procedure under Poisson sampling results in inefficient use of the data and offers a non-trivial chance to accept the null hypothesis regardless of the data. Second, critical values for the CI test by Canonne et al. [op. cit.] involve unknown constants, which make it difficult to control the type I error. While this article considers linear combinations of fourth-order U-statistics as test statistics like Canonne et al. [op. cit.], it employs permutation methods via Monte Carlo sampling to calibrate test statistics.

Furthermore, classical χ^2- and G-tests are commonly applied for discrete CI testing. However, these are asymptotic tests, and thus their finite-sample validity is unclear. Inevitably, miscalibration of these test statistics could be even more serious for a large d. Indeed, it is demonstrated in this article that these tests have asymptotically zero power (and thus become sub-optimal) under some high-dimensional setup. The practical value of the proposed CI tests and sub-optimality of classical tests are confirmed in the extensive simulation study, and the proposed tests are also applied to real-world datasets.

Masayuki Hirukawa

[References]

1. Agresti, A. (1992). A survey of exact inference for contingency tables. *Statistical Science* **7** 131–153. MR1173420 MR1173420
2. Agresti, A. (2003). *Categorical Data Analysis*. John Wiley & Sons. MR1914507 MR1914507
3. Arias-Castro, E., Pelletier, B. and Saligrama, V. (2018). Remember the curse of dimensionality: The case of goodness-of-fit testing in arbitrary dimension. *Journal of Nonparametric Statistics* **30** 448–471. MR3794401 MR3794401
4. Balakrishnan, S. and Wasserman, L. (2018). Hypothesis testing for high-dimensional multinomials: A selective review. *The Annals of Applied Statistics* **12** 727–749. MR3834283 MR3834283
5. Balakrishnan, S. and Wasserman, L. (2019). Hypothesis testing for densities and high-dimensional multinomials: Sharp local minimax rates. *The Annals of Statistics* **47** 1893–1927. MR3953439 MR3953439
6. Berrett, T. B., Kontoyiannis, I. and Samworth, R. J. (2021). Optimal rates for independence testing via U-statistic permutation tests. *The Annals of Statistics* **49** 2457–2490. MR4338371 MR4338371
7. Berrett, T. B. and Samworth, R. J. (2021). USP: an independence test that

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improves on Pearson's chi-squared and the G-test. *Proceedings of the Royal Society A* **477** 20210549. MR4366500 MR4366500

8. Berrett, T. B., Wang, Y., Barber, R. F. and Samworth, R. J. (2020). The conditional permutation test for independence while controlling for confounders. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **82** 175–197. MR4060981 MR4060981
9. Bickel, P. J., Hammel, E. A. and O'Connell, J. W. (1975). Sex Bias in Graduate Admissions: Data from Berkeley: Measuring bias is harder than is usually assumed, and the evidence is sometimes contrary to expectation. *Science* **187** 398–404.
10. Bishop, Y. M., Fienberg, S. E. and Holland, P. W. (2007). *Discrete Multivariate Analysis: Theory and Practice*. Springer Science & Business Media. MR2344876 MR2344876
11. Boucheron, S., Lugosi, G. and Massart, P. (2013). *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press. MR3185193 MR3185193
12. Candés, E., Fan, Y., Janson, L. and Lv, J. (2018). Panning for gold: 'model-X' knockoffs for high dimensional controlled variable selection. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **80** 551–577. MR3798878 MR3798878
13. Canonne, C. L., Diakonikolas, I., Kane, D. M. and Stewart, A. (2018). Testing conditional independence of discrete distributions. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing. STOC 2018* 735–748. Association for Computing Machinery, New York, NY, USA. MR3826290 MR3826290
14. Canonne, C. L. et al. (2022). Topics and techniques in distribution testing: a biased but representative sample. *Foundations and Trends® in Communications and Information Theory* **19** 1032–1198.
15. Chan, S.-O., Diakonikolas, I., Valiant, P. and Valiant, G. (2014). Optimal algorithms for testing closeness of discrete distributions. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms* 1193–1203. SIAM. MR3376448 MR3376448
16. Chatterjee, S. (2022). A survey of some recent developments in measures of association. *arXiv preprint arXiv:2211.04702*. MR4824375
17. Cormen, T. H., Leiserson, C. E., Rivest, R. L. and Stein, C. (2022). *Introduction to Algorithms*. MIT Press. MR2572804 MR2572804
18. De Campos, L. M. and Huete, J. F. (2000). A new approach for learning belief networks using independence criteria. *International Journal of Approximate Reasoning* **24** 11–37. MR1759736 MR1759736
19. Diakonikolas, I. and Kane, D. M. (2016). A new approach for testing properties of discrete distributions. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)* 685–694. IEEE. MR3631031 MR3631031
20. Gretton, A., Bousquet, O., Smola, A. and Schölkopf, B. (2005). Measuring statistical dependence with Hilbert-Schmidt norms. In *International Conference on Algorithmic Learning Theory* 63–77. Springer. MR2255909 MR2255909
21. Haberman, S. J. (1988). A warning on the use of chi-squared statistics with frequency tables with small expected cell counts. *Journal of the American Statistical Association* **83** 555–560. MR0971386 MR0971386
22. Imbens, G. W. and Rubin, D. B. (2015). *Causal Inference in Statistics, Social, and Biomedical Sciences*. Cambridge University Press. MR3309951 MR3309951
23. Jacquet, P. and Szpankowski, W. (1998). Analytical depoissonization and its

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applications. *Theoretical Computer Science* **201** 1–62. MR1625392
24. Joag-Dev, K. and Proschan, F. (1983). Negative association of random variables with applications. *The Annals of Statistics* **11** 286–295. MR0684886 MR0684886
25. Kac, M. (1949). On deviations between theoretical and empirical distributions. *Proceedings of the National Academy of Sciences* **35** 252–257. MR0029490 MR0029490
26. Kim, I. (2020). Multinomial goodness-of-fit based on U-statistics: Highdimensional asymptotic and minimax optimality. *Journal of Statistical Planning and Inference* **205** 74–91. MR4011624 MR4011624
27. Kim, I. (2022). Comments on "Testing conditional independence of discrete distributions". *arXiv preprint arXiv:2207.02819*.
28. Kim, I., Balakrishnan, S. and Wasserman, L. (2022). Minimax optimality of permutation tests. *The Annals of Statistics* **50** 225–251. MR4382015 MR4382015
29. Kim, I., Neykov, M., Balakrishnan, S. and Wasserman, L. (2022). Local permutation tests for conditional independence. *The Annals of Statistics* **50** 3388–3414. MR4524501 MR4524501
30. Kim, I., Neykov, M., Balakrishnan, S. and Wasserman, L. (2024). Supplement to "Conditional independence testing for discrete distributions: Beyond χ^2- and G-tests". https://doi.org/10.1214/24-EJS2315SUPP.
31. Koller, D. and Friedman, N. (2009). *Probabilistic Graphical Models: Principles and Techniques*. MIT Press. MR2816736 MR2816736
32. Li, C. and Fan, X. (2020). On nonparametric conditional independence tests for continuous variables. *Wiley Interdisciplinary Reviews: Computational Statistics* **12** e1489. MR4098477 MR4098477
33. Liu, M., Katsevich, E., Janson, L. and Ramdas, A. (2022). Fast and powerful conditional randomization testing via distillation. *Biometrika* **109** 277–293. MR4430958 MR4430958
34. McDonald, J. H. (2014). G-test of goodness-of-fit. *Handbook of Biological Statistics* **487** 53–58.
35. Neykov, M., Balakrishnan, S. and Wasserman, L. (2021). Minimax optimal conditional independence testing. *The Annals of Statistics* **49** 2151–2177. MR4319245 MR4319245
36. Neykov, M., Wasserman, L., Kim, I. and Balakrishnan, S. (2023). Nearly minimax optimal Wasserstein conditional independence testing. *arXiv preprint arXiv:2308.08672*. MR4319245 MR4838451
37. Pearl, J. (2014). *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Elsevier. MR0965765 MR0965765
38. Pearson, K. (1900). On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **50** 157–175.
39. Penrose, M. (2003). *Random Geometric Graphs* **5**. OUP Oxford. MR1986198 MR1986198
40. Petersen, L. and Hansen, N. R. (2021). Testing conditional independence via quantile regression based partial copulas. *Journal of Machine Learning Research* **22** 1–47. MR4253763 MR4253763
41. Romano, J. P. and Wolf, M. (2005). Exact and approximate stepdown methods for multiple hypothesis testing. *Journal of the American Statistical Association* **100** 94–108. MR2156821 MR2156821
42. Shah, R. D. and Peters, J. (2020). The hardness of conditional independence testing and the generalised covariance measure. *The Annals of Statistics* **48** 1514–1538. MR4124333 MR4124333

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43. Song, L., Smola, A., Gretton, A., Bedo, J. and Borgwardt, K. (2012). Feature selection via dependence maximization. *Journal of Machine Learning Research* **13** 1393–1434. MR2930643 MR2930643
44. Spohn, W. (1994). On the properties of conditional independence. In *Patrick Suppes: Scientific Philosopher* 173–196. Springer. MR1349350 MR1349350
45. Tsamardinos, I. and Borboudakis, G. (2010). Permutation testing improves Bayesian network learning. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases* 322–337. Springer.
46. Vaart, A. W. and Wellner, J. A. (1996). Weak convergence. In *Weak Convergence and Empirical Processes* 16–28. Springer. MR1385671 MR1385671
47. Valiant, G. and Valiant, P. (2017). An automatic inequality prover and instance optimal identity testing. *SIAM Journal on Computing* **46** 429–455. MR3614697 MR3614697
48. Valiant, P. (2011). Testing symmetric properties of distributions. *SIAM Journal on Computing* **40** 1927–1968. MR2863200 MR2863200
49. Yao, Q. and Tritchler, D. (1993). An exact analysis of conditional independence in several 2×2 contingency tables. *Biometrics* 233–236. MR1221407 MR1221407
50. Zhang, J. and Zhang, Z. (2024). A normal test for independence via generalized mutual information. *Statistics & Probability Letters* **210** 110113. MR4723470 MR4723470

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.