Supplement to "Nonparametric Estimation of Splicing Points in Skewed Cost Distributions: A Kernel-Based Approach"

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8th April 2025

S.1 Introduction

In the proofs of lemmata below, equation numbers starting from "A" correspond to those in the Appendix. Before proceeding, we reproduce Appendix A.1 and A.2 as S.1.1 and S.1.2, respectively, for convenience.

S.2 Useful Formulae on the Gamma Function

Stirling's Formula.

$$\Gamma(a+1) = \sqrt{2\pi} a^{a+1/2} \exp(-a) \left\{ 1 + \frac{1}{12a} + O\left(a^{-2}\right) \right\} \text{ as } a \to \infty. \tag{S1}$$

Recursive Formula for the Lower Incomplete Gamma Function.

$$\gamma(a+1, z) = a\gamma(a, z) - z^a \exp(-z) \text{ for } a, z > 0. \tag{S2}$$

Recursive Formula for the Polygamma Function.

$$\Psi^{(m)}(a+1) = \Psi^{(m)}(a) + \frac{(-1)^m m!}{a^{m+1}} \text{ for } a > 0 \text{ and } m \in \{0, 1, 2, ...\}. \tag{S3}$$

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S.3 Additional Notation

The following notation is adopted in the proofs: $R(a,z) = z^a \exp(-z)/\Gamma(a+1)$ for $a, z > 0$; $\dot{K}_c^{\pm}(u) = \partial K_x^{\pm}(u)/\partial x|_{x=c}$; $\ddot{K}_c^{\pm}(u) = \partial^2 K_x^{\pm}(u)/\partial x^2|_{x=c}$; $\dddot{K}_c^{\pm}(u) = \partial^3 K_x^{\pm}(u)/\partial x^3|_{x=c}$; $H_i = \dot{K}_{t_0}^{-}(X_i) - \dot{K}_{t_0}^{+}(X_i)$; and $z^{\pm} = (t_0 \pm \Delta)/b$.

S.4 Proofs of Lemmata

S.4.1 Proof of Lemma A1

First two statements are direct outcomes from the proof of Lemma A.3 in Funke and Hirukawa (2024). The last statement can be obtained by equation (2.5) of Guo and Qi (2010) and (S1). ■

S.4.2 Proof of Lemma A2

The proof closely follows the one of Lemma 3(ii) in Funke and Hirukawa (2025). However, we provide the proof in full in order to make this supplement self-contained.

In what follows, the cases of $u = 0$ and $u > 0$ are considered separately. For $u = 0$, it suffices to show that $\lim_{u\uparrow 0} \dot{K}_x^{\pm}(u) = \lim_{u\downarrow 0} \dot{K}_x^{\pm}(u) = 0$. If this is the case, then $\dot{K}_x^{\pm}(u) = 0$ and the result trivially holds. The zero left limit can be immediately established by $K_x^{\pm}(u) \equiv 0$ for $u < 0$. To evaluate the right limit, observe that for $u > 0$,

$$\begin{aligned}\dot{K}_x^{\pm}(u) &= \frac{1}{b}\left\{\ln u - \ln b - \Psi\left(a^{\pm}+1\right)\right\} K_x^{\pm}(u) \\ &= \frac{1}{b}\left\{(\ln u) K_x^{\pm}(u) - \left\{\ln b + \Psi\left(a^{\pm}+1\right)\right\} K_x^{\pm}(u)\right\}. \end{aligned} \tag{S4}$$

It follows from $a^{\pm} > 0$ that $\lim_{u\downarrow 0} K_x^{\pm}(u) = 0$. By L'Hôpital's rule, $\lim_{u\downarrow 0} (\ln u) K_x^{\pm}(u) = 0$ also holds. Hence, the right limit is shown to be zero.

For $u > 0$, it follows from (S4) that

$$\left|\dot{K}_x^{\pm}(u)\right| \le \frac{1}{b}\left\{|\ln u| K_x^{\pm}(u) + \left|\ln b + \Psi\left(a^{\pm}+1\right)\right| K_x^{\pm}(u)\right\}. \tag{S5}$$

We work on the uniform bound of $|\ln b + \Psi(a^{\pm}+1)| K_x^{\pm}(u)$ first. Observe that $|\ln b + \Psi(a^{\pm}+1)| = |\ln x| + o(1)$, where the $o(1)$ approximation error is uniform on I_0 by Lemma A1. It is also the case that $|\ln x| \le \max\{x^{-1}, x\}$, and we may take the $o(1)$ term no greater than 1 for a sufficiently large n. Then, $|\ln b + \Psi(a^{\pm}+1)| \le \max\{x^{-1}, x\} + 1$. It follows from $(x \pm \Delta)^{-1/2} = x^{-1/2}\{1 + o(1)\}$ and $o(1) \le 1$ for a sufficiently large n in (A6) that

$$\left|\ln b + \Psi\left(a^{\pm}+1\right)\right| K_x^{\pm}(u) \leq \frac{b^{-1/2}\{1+o(1)\}}{\sqrt{2\pi}}\left[\max\left\{x^{-3/2}, x^{1/2}\right\} + x^{-1/2}\right]$$
$$\leq b^{-1/2}\sqrt{\frac{2}{\pi}}\left[\max\left\{\underline{t}^{-3/2}, \bar{t}^{1/2}\right\} + \underline{t}^{-1/2}\right] \tag{S6}$$

The relation $|\ln u| \leq \max\{u^{-1}, u\}$ again helps find the uniform bound of $|\ln u| K_x^{\pm}(u)$. Suppose that $|\ln u| \leq u^{-1}$. Because $u^{-1}K_x^{\pm}(u)$ is maximized at $u = x \pm \Delta - b$, it holds that $|\ln u| K_x^{\pm}(u) \leq u^{-1}K_x^{\pm}(u) \leq (x \pm \Delta - b)^{-1} K_x^{\pm}(x \pm \Delta - b)$. By (S1),

$$(x \pm \Delta - b)^{-1} K_x^{\pm}(x \pm \Delta - b)$$
$$= \frac{b^{-1/2}e\{1+o(1)\}}{\sqrt{2\pi}}\left(1 - \frac{1}{a^{\pm}}\right)^{a^{\pm}} \frac{1}{(x \pm \Delta - b)\sqrt{x \pm \Delta}},$$

where

$$\left(1 - \frac{1}{a^{\pm}}\right)^{a^{\pm}} = e^{-1}\{1+o(1)\} \text{ and } \frac{1}{(x \pm \Delta - b)\sqrt{x \pm \Delta}} = x^{-3/2}\{1+o(1)\}.$$

It follows from $x^{-3/2} \leq \underline{t}^{-3/2}$ and $o(1) \leq 1$ for a sufficiently large n that

$$|\ln u| K_x^{\pm}(u) \leq \frac{b^{-1/2}\{1+o(1)\}}{\sqrt{2\pi}x^{3/2}} \leq b^{-1/2}\sqrt{\frac{2}{\pi}}\underline{t}^{-3/2}.$$

Alternatively, suppose that $|\ln u| \leq u$. In this case, $uK_x^{\pm}(u)$ is maximized at $u = x \pm \Delta + b$, and thus $|\ln u| K_x^{\pm}(u) \leq uK_x^{\pm}(u) \leq (x \pm \Delta + b)^{-1} K_x^{\pm}(x \pm \Delta + b)$. Again by (S1),

$$(x \pm \Delta + b)^{-1} K_x^{\pm}(x \pm \Delta + b)$$
$$= \frac{b^{-1/2}e^{-1}\{1+o(1)\}}{\sqrt{2\pi}}\left(1 + \frac{1}{a^{\pm}}\right)^{a^{\pm}} \frac{x \pm \Delta + b}{\sqrt{x \pm \Delta}},$$

where

$$\left(1 + \frac{1}{a^{\pm}}\right)^{a^{\pm}} = e\{1+o(1)\} \text{ and } \frac{x \pm \Delta + b}{\sqrt{x \pm \Delta}} = \sqrt{x}\{1+o(1)\}.$$

Therefore,

$$|\ln u| K_x^{\pm}(u) \leq \frac{b^{-1/2}\sqrt{x}\{1+o(1)\}}{\sqrt{2\pi}} \leq b^{-1/2}\sqrt{\frac{2}{\pi}}\bar{t}^{1/2},$$

by $x^{1/2} \leq \bar{t}^{1/2}$ and again $o(1) \leq 1$ for a sufficiently large n. Combining these two scenarios, we have

$$|\ln u| K_x^{\pm}(u) \leq \max\left\{u^{-1}K_x^{\pm}(u), uK_x^{\pm}(u)\right\} \leq b^{-1/2}\sqrt{\frac{2}{\pi}}\max\left\{\underline{t}^{-3/2}, \bar{t}^{1/2}\right\}. \tag{S7}$$

In the end, substituting (S6) and (S7) into (S5) yields

$$\sup_{(x,u)\in I_0\times\mathbb{R}_+}\left|\dot{K}_x^{\pm}(u)\right| \le b^{-3/2}\sqrt{\frac{2}{\pi}}\left[2\max\left\{\underline{t}^{-3/2},\bar{t}^{1/2}\right\}+\underline{t}^{-1/2}\right] =: b^{-3/2}\sqrt{\frac{2}{\pi}}C_0.$$

This completes the proof. ■

S.4.3 Proof of Lemma A4

The lemma is proven by induction. The case for $m=0$ is obvious by (S2) and $\Gamma(a+1)=a\Gamma(a)$. Next, suppose that the statement holds for some $m\geq 0$. Then,

$$\begin{aligned}
&\lambda^{m+1}\frac{\gamma(a+m+2,z)}{\Gamma(a+1)}\\
&=\lambda^{m+1}\frac{(a+m+1)\gamma(a+m+1,z)-z^{a+m+1}\exp(-z)}{\Gamma(a+1)}\\
&=\lambda(a+m+1)\{p_mP(a,z)-r_mR(a,z)\}-(\lambda z)^{m+1}R(a,z)\\
&=\{\lambda(a+m+1)p_m\}P(a,z)-\left\{\lambda(a+m+1)r_m+(\lambda z)^{m+1}\right\}R(a,z),
\end{aligned}$$

where the first and second equalities come from (S2) and the assumption of induction, respectively. The statement for $m+1$ is established by the correspondence of coefficients on $P(a,z)$ and $R(a,z)$. ■

S.4.4 Proof of Lemma A5

The first statement can be obtained by setting $x=t_0$ in (A15) and then substituting $\Phi(0)=1/2$, $\phi(0)=1/\sqrt{2\pi}$ and $\phi^{(1)}(0)=0$. For the second statement, putting $x=t_0$ in (A17) gives

$$R\left(z^{\pm},z_0\right)=\left\{\frac{1+O\left(1/z^{\pm}\right)}{\sqrt{2\pi}}\right\}\frac{1}{\sqrt{z^{\pm}}}\exp\left\{z^{\pm}\left(\ln\xi^{\pm}+1-\xi^{\pm}\right)\right\},$$

for $\xi^{\pm}:=z_0/z^{\pm}=t_0/(t_0\pm\Delta)$. When $x=t_0$, (A16) reduces to

$$\frac{1}{\sqrt{z^{\pm}}}=\frac{b^{1/2}}{\sqrt{t_0}}\left\{1\mp\frac{\Delta}{2t_0}+O\left(\Delta^2\right)\right\}.$$

Moreover, $\exp\left\{z^{\pm}\left(\ln\xi^{\pm}+1-\xi^{\pm}\right)\right\}$ can be approximated by

$$\exp\left\{z^{\pm}\left(\ln\xi^{\pm}+1-\xi^{\pm}\right)\right\}=1-\frac{1}{2t_0}\left(\frac{\Delta^2}{b}\right)+O\left(\frac{\Delta^3}{b}\right).$$

Then, the second statement can be obtained by recognizing that $1+O\left(1/z^{\pm}\right)=1+O(b)$. ■

S.4.5 Proof of Lemma A6

Observe that

$$E\left\{\dot{K}_{t_0}^{\pm}\left(X_i\right)\right\}=\int_0^{\infty}\dot{K}_{t_0}^{\pm}\left(u\right)g\left(u\right)du+d_0\int_0^{t_0}\dot{K}_{t_0}^{\pm}\left(u\right)du=:E_g^{\pm}+E_{d_0}^{\pm},$$

where

$$\dot{K}_{t_0}^{\pm}\left(u\right)=\frac{1}{b}\left\{\ln u-\ln b-\Psi\left(z^{\pm}+1\right)\right\}K_{t_0}^{\pm}\left(u\right).$$

Then,

$$E\left\{\hat{J}^{(1)}\left(t_0\right)\right\}=E\left\{\dot{K}_{t_0}^{-}\left(X_i\right)\right\}-E\left\{\dot{K}_{t_0}^{+}\left(X_i\right)\right\}=\left(E_g^{-}-E_g^{+}\right)+\left(E_{d_0}^{-}-E_{d_0}^{+}\right).$$

We start from approximating $\left(E_g^{-}-E_g^{+}\right)$ and then proceed to $\left(E_{d_0}^{-}-E_{d_0}^{+}\right)$. A minor modification of the proof for Theorem 2.1(i) of Funke and Hirukawa (2024) yields

$$E_g^{\pm}=g^{(1)}\left(t_0\right)\pm g^{(2)}\left(t_0\right)\Delta+O\left(b\right).$$

It follows that

$$E_g^{-}-E_g^{+}=-2g^{(2)}\left(t_0\right)\Delta+O\left(b\right).$$

Next, by Lemma A1 and a second-order Taylor expansion of $\ln u$ around $u=t_0$,

$$\begin{aligned}\ln u-\ln b-\Psi\left(z^{\pm}+1\right)&=\ln u-\ln t_0\mp\frac{\Delta}{t_0}-\frac{b}{2t_0}+O\left(b^2\right)\\&\sim\left(\frac{u-t_0}{t_0}\right)-\frac{1}{2}\left(\frac{u-t_0}{t_0}\right)^2\mp\frac{\Delta}{t_0}-\frac{b}{2t_0}+O\left(b^2\right).\end{aligned}$$

Then,

$$\begin{aligned}E_{d_0}^{\pm}\sim\frac{d_0}{b}&\left[\int_0^{t_0}\left(\frac{u-t_0}{t_0}\right)K_{t_0}^{\pm}\left(u\right)du-\frac{1}{2}\int_0^{t_0}\left(\frac{u-t_0}{t_0}\right)^2K_{t_0}^{\pm}\left(u\right)du\right.\\&\left.+\left\{\mp\frac{\Delta}{t_0}-\frac{b}{2t_0}+O\left(b^2\right)\right\}\int_0^{t_0}K_{t_0}^{\pm}\left(u\right)du\right].\end{aligned}$$

Also observe that

$$\int_0^{t_0}u^mK_{t_0}^{\pm}\left(u\right)du=:b^m\frac{\gamma\left(z^{\pm}+m+1,z_0\right)}{\Gamma\left(z^{\pm}+1\right)}$$

for $m\geq 0$. Then, by Lemma A4 and straightforward calculations, we can obtain

$$\begin{aligned}&\int_0^{t_0}K_{t_0}^{\pm}\left(u\right)du=P\left(z^{\pm},z_0\right)-R\left(z^{\pm},z_0\right),\\&\int_0^{t_0}\left(\frac{u-t_0}{t_0}\right)K_{t_0}^{\pm}\left(u\right)du=\left(\pm\frac{\Delta}{t_0}+\frac{b}{t_0}\right)P\left(z^{\pm},z_0\right)-\left(1\pm\frac{\Delta}{t_0}+\frac{b}{t_0}\right)R\left(z^{\pm},z_0\right),\end{aligned}$$

and

$$\int_0^{t_0}\left(\frac{u-t_0}{t_0}\right)^2 K_{t_0}^{\pm}(u)\,du$$
$$=\left(\frac{b}{t_0}+\frac{2b^2}{t_0^2}\pm\frac{3\Delta b}{t_0^2}+\frac{\Delta^2}{t_0^2}\right)P\left(z^{\pm},z_0\right)-\left(\frac{3b}{t_0}\pm\frac{\Delta}{t_0}+\frac{2b^2}{t_0^2}\pm\frac{3\Delta b}{t_0^2}+\frac{\Delta^2}{t_0^2}\right)R\left(z^{\pm},z_0\right).$$

It follows that

$$E_{d_0}^{\pm}\sim\frac{d_0}{b}\left[\left\{\mp\frac{3\Delta b}{2t_0^2}+O\left(\Delta^2\right)\right\}P\left(z^{\pm},z_0\right)-\left\{1-\frac{b}{t_0}\mp\frac{\Delta}{2t_0}+O\left(\Delta^2\right)\right\}R\left(z^{\pm},z_0\right)\right]$$

Substituting Lemma A5 and then making straightforward but tedious calculations, we have

$$E_{d_0}^{-}-E_{d_0}^{+}=-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right)+O\left(\frac{\Delta^3}{b^{3/2}}\right).$$

In the end,

$$\begin{aligned}E\left\{\hat{J}^{(1)}(t_0)\right\}&=O(\Delta)+\left\{-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right)+O\left(\frac{\Delta^3}{b^{3/2}}\right)\right\}\\&=-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right)+O\left(\frac{\Delta^3}{b^{3/2}}\right)\\&\sim-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right).\end{aligned}$$

This completes the proof. ■

S.4.6 Proof of Lemma A7

Observe that

$$Var\left\{\hat{J}^{(1)}(t_0)\right\}=\frac{1}{n}Var(H_i)=\frac{1}{n}\left\{E\left(H_i^2\right)-E^2(H_i)\right\},$$

where $E(H_i)=E_{d_0}^{-}-E_{d_0}^{+}=O\left(\Delta/b^{1/2}\right)$ as shown above. Below we concentrate on approximating

$$E\left(H_i^2\right)=\int_0^{\infty}\left\{\dot{K}_{t_0}^{-}(u)-\dot{K}_{t_0}^{+}(u)\right\}^2 f(u)\,du.$$

A difficulty arises at this stage because of asymmetry in $\dot{K}_x^{-}(\cdot)-\dot{K}_x^{+}(\cdot)$. If it were symmetric, we could first approximate the right-hand side by

$$\left\{\frac{f(t_0^{-})+f(t_0^{+})}{2}\right\}\int_0^{\infty}\left\{\dot{K}_{t_0}^{-}(u)-\dot{K}_{t_0}^{+}(u)\right\}^2 du.$$

However, it is not obvious whether this type of manipulation is still valid for the asymmetric case. Instead, our proof strategy is to decompose $f(\cdot)$ into two components so that $E(H_i^2) =: V_g + V_{d_0}$, where

$$V_g = \int_0^\infty \left\{ \dot{K}_{t_0}^- (u) - \dot{K}_{t_0}^+ (u) \right\}^2 g(u)\, du, \text{ and}$$
$$V_{d_0} = d_0 \int_0^{t_0} \left\{ \dot{K}_{t_0}^- (u) - \dot{K}_{t_0}^+ (u) \right\}^2 du.$$

Below it is demonstrated that two integrals can be individually approximated by

$$V_g \sim g(t_0) \frac{3}{2\sqrt{\pi} t_0^{5/2}} \left(\frac{\Delta^2}{b^{5/2}} \right), \text{ and} \tag{S8}$$
$$V_{d_0} \sim \left(\frac{d_0}{2} \right) \frac{3}{2\sqrt{\pi} t_0^{5/2}} \left(\frac{\Delta^2}{b^{5/2}} \right). \tag{S9}$$

It should be recognized that (S8) and (S9) jointly establish the lemma, because $g(t_0) + d_0/2 = \left\{ f(t_0^-) + f(t_0^+) \right\} /2$.

S.4.6.1 Proof of (S8)

Define $V_g^{2\pm} := \int_0^\infty \left\{ \dot{K}_{t_0}^\pm (u) \right\}^2 g(u)\, du$ and $V_g^{+-} := \int_0^\infty \dot{K}_{t_0}^- (u)\, \dot{K}_{t_0}^+ (u)\, g(u)\, du$. Then, V_g can be decomposed into $V_g = V_g^{2-} + V_g^{2+} - 2V_g^{+-}$. In what follows, approximations to $V_g^{2\pm}$ and V_g^{+-} are derived separately, and (S8) can be obtained in the end.

(i) Approximation to $V_g^{2\pm}$. For $Y^\pm \overset{d}{=} G(2z^\pm + 1, b/2)$ and $A^\pm(x)$ defined in (A7), we can express $V_g^{2\pm}$ as

$$\begin{aligned} V_g^{2\pm} &= \frac{1}{b^2} \int_0^\infty \left\{ \ln u - \ln b - \Psi\left(z^\pm + 1\right) \right\}^2 \left\{ K_{t_0}^\pm (u) \right\}^2 g(u)\, du \\ &= \frac{A^\pm(t_0)}{b^2} E\left[\left\{ \ln Y^\pm - \ln b - \Psi\left(z^\pm + 1\right) \right\}^2 g\left(Y^\pm\right) \right]. \end{aligned} \tag{S10}$$

By (S1) and $b = o(\Delta)$, $A^\pm(t_0)$ can be approximated by

$$\begin{aligned} A^\pm(t_0) &= \frac{b^{-1} \Gamma(2z^\pm + 1)}{2^{2z^\pm + 1} \Gamma^2 (z^\pm + 1)} \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_0 \pm \Delta}} \{1 + O(b)\} \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_0}} \{1 + O(\Delta)\}. \end{aligned} \tag{S11}$$

In addition, a mean-value expansion of $g(Y^{\pm})$ around $Y^{\pm}=t_0$ yields

$$\begin{aligned}
& E\left[\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2 g\left(Y^{\pm}\right)\right] \\
= \; & g\left(t_0\right) E\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2 \\
& +E\left[\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2 g^{(1)}\left(\bar{t}_0^{\pm}\right)\left(Y^{\pm}-t_0\right)\right] \\
=: \; & W_1^{\pm}+W_2^{\pm}
\end{aligned}$$

for some $\bar{t}_0^{\pm}$ on the line segment joining $Y^{\pm}$ and t_0. For $W_1^{\pm}$, it follows from the proof for Theorem 2.1(ii) of Funke and Hirukawa (2024) and Lemma A1 that

$$\begin{aligned}
& E\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2 \\
& =\left\{\ln 2+\Psi\left(z^{\pm}+1\right)-\Psi\left(2 z^{\pm}+1\right)\right\}^2+\Psi^{(1)}\left(2 z^{\pm}+1\right) \\
& =\left\{\frac{b}{4\left(t_0 \pm \Delta\right)}+O\left(b^2\right)\right\}^2+\left\{\frac{b}{2\left(t_0 \pm \Delta\right)}-\frac{b^2}{8\left(t_0 \pm \Delta\right)^2}+O\left(b^3\right)\right\} \\
& =\frac{b}{2\left(t_0 \pm \Delta\right)}\{1+O(b)\} .
\end{aligned}$$

It follows from $b=o(\Delta)$ that

$$W_1^{\pm}=\frac{b g\left(t_0\right)}{2\left(t_0 \pm \Delta\right)}\{1+O(b)\}=\frac{b g\left(t_0\right)}{2 t_0}\{1+O(\Delta)\} .$$

Next, $\left|W_2^{\pm}\right|=O\left(b^{3/2}\right)$ is demonstrated. By the Cauchy-Schwarz in equality,

$$\begin{aligned}
\left|W_2^{\pm}\right| & \leq \sup_{x \in \mathbb{R}_+}\left|g^{(1)}(x)\right| E\left\{\left|\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right|^2\left|Y^{\pm}-t_0\right|\right\} \\
& \leq \sup_{x \in \mathbb{R}_+}\left|g^{(1)}(x)\right|\left[E\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2\right]^{1/2} \\
& \quad \times\left[E\left\{\left(\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right)^2\left(Y^{\pm}-t_0\right)^2\right\}\right]^{1/2},
\end{aligned}$$

where $\sup_{x \in \mathbb{R}_+}\left|g^{(1)}(x)\right|<\infty$ by Assumption 2(ii), and $E\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2= O(b)$ as shown above. A straightforward extension of Lemma A.1 in Funke and Hirukawa (2024) yields

$$E\left(X^m \ln^2 X\right)=\beta^m \prod_{k=1}^m(\alpha+k-1)\left[\{\ln \beta+\Psi(\alpha+m)\}^2+\Psi^{(1)}(\alpha+m)\right]$$

for $X \stackrel{d}{=} G(\alpha, \beta)$ and $m \geq 1$. Using this and Lemma A1 and making straightforward but tedious calculations, we finally have $E\left\{\left(\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right)^2\left(Y^{\pm}-t_0\right)^2\right\}= O\left(b^2\right)$, which establishes that $\left|W_2^{\pm}\right|=O\left(b^{3/2}\right)$. Therefore, by $\Delta=o\left(b^{1/2}\right)$,

$$E\left[\left\{\ln Y^{\pm}-\ln b-\Psi\left(z^{\pm}+1\right)\right\}^2 g\left(Y^{\pm}\right)\right]=\frac{b g\left(t_0\right)}{2 t_0}\left\{1+O\left(b^{1/2}\right)\right\} . \tag{S12}$$

Substituting (S11) and (S12) into (S10) and using $\Delta = o\left(b^{1/2}\right)$ again, we may conclude that

$$\begin{aligned} V_g^{2\pm} &= \frac{1}{b^2}\left[\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_0}}\left\{1+O\left(\Delta\right)\right\}\right]\left[\frac{bg\left(t_0\right)}{2t_0}\left\{1+O\left(b^{1/2}\right)\right\}\right] \\ &= \frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{g\left(t_0\right)+O\left(b^{1/2}\right)\right\}. \end{aligned} \tag{S13}$$

(ii) Approximation to V_g^{+-}. Let $Y_0 \overset{d}{=} G\left(2z_0+1, b/2\right)$. Then,

$$\begin{aligned} V_g^{+-} &= \frac{B\left(t_0\right)}{b^2}E\left[\left\{\ln Y_0 - \ln b - \Psi\left(z^- +1\right)\right\}\right. \\ &\quad \left.\times\left\{\ln Y_0 - \ln b - \Psi\left(z^+ +1\right)\right\}g\left(Y_0\right)\right], \end{aligned} \tag{S14}$$

where

$$B\left(t_0\right) := \frac{b^{-1}\Gamma\left(2z_0+1\right)}{2^{2z_0+1}\Gamma\left(z^- +1\right)\Gamma\left(z^+ +1\right)}.$$

It follows from (S1) and $b = o\left(\Delta^2/b\right)$ that

$$\begin{aligned} &B\left(t_0\right) \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}}\left(\frac{t_0}{t_0^2-\Delta^2}\right)^{1/2}\left\{\left(\frac{t_0}{t_0-\Delta}\right)^{t_0-\Delta}\left(\frac{t_0}{t_0+\Delta}\right)^{t_0+\Delta}\right\}^{1/b}\left\{1-\frac{b}{8t_0}+o\left(\Delta^2\right)\right\} \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_0}}\left\{1+O\left(\Delta^2\right)\right\}\left\{1-\frac{1}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}\left\{1+O\left(b\right)\right\} \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_0}}\left\{1-\frac{1}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}. \end{aligned} \tag{S15}$$

Moreover, by a mean-value expansion of $g\left(Y_0\right)$ around $Y_0 = t_0$,

$$\begin{aligned} &E\left[\left\{\ln Y_0 - \ln b - \Psi\left(z^- +1\right)\right\}\left\{\ln Y_0 - \ln b - \Psi\left(z^+ +1\right)\right\}g\left(Y_0\right)\right] \\ =\ & g\left(t_0\right)E\left[\left\{\ln Y_0 - \ln b - \Psi\left(z^- +1\right)\right\}\left\{\ln Y_0 - \ln b - \Psi\left(z^+ +1\right)\right\}\right] \\ &+E\left[\left\{\ln Y^\pm - \ln b - \Psi\left(z^\pm +1\right)\right\}^2 g^{(1)}\left(\bar{t}_0\right)\left(Y_0 - t_0\right)\right] \\ =:\ & W_1^0 + W_2^0 \end{aligned}$$

for some $\bar{t}_0$ on the line segment joining Y_0 and t_0. It can be deduced again from the proof for Theorem 2.1(ii) of Funke and Hirukawa (2024) and Lemma A1 that

$$
\begin{aligned}
&E\left[\left\{\ln Y_0-\ln b-\Psi\left(z^-+1\right)\right\}\left\{\ln Y_0-\ln b-\Psi\left(z^++1\right)\right\}\right]\\
&=\left\{\ln 2+\Psi\left(z^-+1\right)-\Psi\left(2z_0+1\right)\right\}\left\{\ln 2+\Psi\left(z^++1\right)-\Psi\left(2z_0+1\right)\right\}\\
&+\Psi^{(1)}\left(2z_0+1\right)\\
&=\left\{-\frac{\Delta}{t_0}+O(b)\right\}\left\{\frac{\Delta}{t_0}+O(b)\right\}+\left\{\frac{b}{2t_0}+O\left(b^2\right)\right\}\\
&=\frac{b}{2t_0}\left\{1-\frac{2}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}.
\end{aligned}
$$

Therefore,

$$
W_1^0=\frac{bg(t_0)}{2t_0}\left\{1-\frac{2}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}.
$$

It can be also shown that $|W_2^0|=O\left(b^{3/2}\right)$, and thus

$$
\begin{aligned}
&E\left[\left\{\ln Y_0-\ln b-\Psi\left(z^-+1\right)\right\}\left\{\ln Y_0-\ln b-\Psi\left(z^++1\right)\right\}g\left(Y_0\right)\right]\\
=&\frac{bg(t_0)}{2t_0}\left\{1-\frac{2}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)+O\left(b^{1/2}\right)\right\}\\
=&\frac{bg(t_0)}{2t_0}\left\{1-\frac{2}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \qquad \text{(S16)}
\end{aligned}
$$

by $b^{1/2}=o\left(\Delta^2/b\right)$.

Substituting (S15) and (S16) into (S14) finally yields

$$
\begin{aligned}
V_g^{+-}&=\frac{1}{b^2}\left[\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_0}}\left\{1-\frac{1}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}\right]\\
&\quad\times\left[\frac{bg(t_0)}{2t_0}\left\{1-\frac{2}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}\right]\\
&=\frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{g(t_0)-\frac{3g(t_0)}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}. \qquad \text{(S17)}
\end{aligned}
$$

(iii) Proof of (S8). By (S13), (S17) and $b^{1/2}=o\left(\Delta^2/b\right)$,

$$
\begin{aligned}
V_g&=\left(V_g^{2-}+V_g^{2+}\right)-2V_{g_Y}^{+-}\\
&=\frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{2g(t_0)+O\left(b^{1/2}\right)\right\}-\frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{2g(t_0)-\frac{6g(t_0)}{t_0}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}\\
&\sim g(t_0)\frac{3}{2\sqrt{\pi}t_0^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right).
\end{aligned}
$$

Therefore, (S8) is established.

S.4.6.2 Proof of (S9)

Denote $V_{d_0}^{2\pm} := d_0 \int_0^{t_0} \left\{ \dot{K}_{t_0}^{\pm}(u) \right\}^2 du$ and $V_{d_0}^{+-} := d_0 \int_0^{t_0} \dot{K}_{t_0}^{-}(u)\, \dot{K}_{t_0}^{+}(u)\, du$ so that $V_{d_0} = V_{d_0}^{2-} + V_{d_0}^{2+} - 2V_{d_0}^{+-}$. As above, $V_{d_0}^{2\pm}$ and $V_{d_0}^{+-}$ are approximated separately.

(i) Approximation to $V_{d_0}^{2\pm}$. Observe that

$$V_{d_0}^{2\pm} = \frac{d_0 A^{\pm}(t_0)}{b^2} \int_0^{t_0} \left\{ \ln u - \ln b - \Psi\left(z^{\pm}+1\right) \right\}^2 \frac{u^{2z^{\pm}} \exp\left\{-u/(b/2)\right\}}{(b/2)^{2z^{\pm}+1}\, \Gamma\left(2z^{\pm}+1\right)} du. \quad \text{(S18)}$$

Lemma A1 implies that

$$\begin{aligned}
&\left\{ \ln u - \ln b - \Psi\left(z^{\pm}+1\right) \right\}^2 \\
&= \left\{ \ln u - \ln t_0 \mp \frac{\Delta}{t_0} - \frac{b}{2t_0} + O\left(\Delta^2\right) \right\}^2 \\
&= \ln^2 u - 2\left\{ \ln t_0 \pm \frac{\Delta}{t_0} + \frac{b}{2t_0} + O\left(\Delta^2\right) \right\} \ln u + \left\{ \ln t_0 \pm \frac{\Delta}{t_0} + \frac{b}{2t_0} + O\left(\Delta^2\right) \right\}^2.
\end{aligned}$$

Combining this with a second-order Taylor expansion of $\ln^m u$ for $m \geq 1$ around $u = t_0$, i.e.,

$$\begin{aligned}
\ln^m u &= \ln^m t_0 + m \ln^{m-1} t_0 \left(\frac{u - t_0}{t_0} \right) \\
&\quad + \frac{m}{2} \left\{ (m-1) \ln^{m-2} t_0 - \ln^{m-1} t_0 \right\} \left(\frac{u - t_0}{t_0} \right)^2 + o\left(|u - t_0|^2 \right), \quad \text{(S19)}
\end{aligned}$$

we have

$$\begin{aligned}
&\left\{ \ln u - \ln b - \Psi\left(z^{\pm}+1\right) \right\}^2 \\
&\sim \left\{ \mp \frac{2\Delta}{t_0} - \frac{b}{t_0} + O\left(\Delta^2\right) \right\} \left(\frac{u - t_0}{t_0} \right) + \left\{ 1 \pm \frac{\Delta}{t_0} + \frac{b}{2t_0} + O\left(\Delta^2\right) \right\} \left(\frac{u - t_0}{t_0} \right)^2 \\
&+ \left\{ \pm \frac{\Delta}{t_0} + \frac{b}{2t_0} + O\left(\Delta^2\right) \right\}^2.
\end{aligned}$$

Observe that

$$\int_0^{t_0} u^m \frac{u^{2z^{\pm}} \exp\left\{-u/(b/2)\right\}}{(b/2)^{2z^{\pm}+1}\, \Gamma\left(2z^{\pm}+1\right)} du = \left(\frac{b}{2} \right)^m \frac{\gamma\left(2z^{\pm}+m+1, 2z_0\right)}{\Gamma\left(2z^{\pm}+1\right)}$$

for $m \geq 0$. Then, Lemma A4 and straightforward calculations lead to

$$\int_0^{t_0} \frac{u^{2z^\pm} \exp\{-u/(b/2)\}}{(b/2)^{2z^\pm+1}\Gamma(2z^\pm+1)} du$$
$$= P\left(2z^\pm, 2z_0\right) - R\left(2z^\pm, 2z_0\right),$$
$$\int_0^{t_0} \left(\frac{u-t_0}{t_0}\right) \frac{u^{2z^\pm} \exp\{-u/(b/2)\}}{(b/2)^{2z^\pm+1}\Gamma(2z^\pm+1)} du$$
$$= \left(\pm\frac{\Delta}{t_0} + \frac{b}{2t_0}\right) P\left(2z^\pm, 2z_0\right) - \left(1 \pm \frac{\Delta}{t_0} + \frac{b}{2t_0}\right) R\left(2z^\pm, 2z_0\right),$$

and

$$\int_0^{t_0} \left(\frac{u-t_0}{t_0}\right)^2 \frac{u^{2z^\pm} \exp\{-u/(b/2)\}}{(b/2)^{2z^\pm+1}\Gamma(2z^\pm+1)} du$$
$$= \left(\frac{b}{2t_0} + \frac{b^2}{2t_0^2} \pm \frac{3\Delta b}{2t_0^2} + \frac{\Delta^2}{t_0^2}\right) P\left(2z^\pm, 2z_0\right)$$
$$- \left(\frac{3b}{2t_0} \pm \frac{\Delta}{t_0} + \frac{b^2}{2t_0^2} \pm \frac{3\Delta b}{2t_0^2} + \frac{\Delta^2}{t_0^2}\right) R\left(2z^\pm, 2z_0\right).$$

It follows that

$$\int_0^{t_0} \left\{\ln u - \ln b - \Psi\left(z^\pm+1\right)\right\}^2 \frac{u^{2z^\pm} \exp\{-u/(b/2)\}}{(b/2)^{2z^\pm+1}\Gamma(2z^\pm+1)} du$$
$$\sim \left\{\frac{b}{2t_0} \pm \frac{\Delta b}{t_0^2} + O\left(\Delta^2\right)\right\} P\left(2z^\pm, 2z_0\right)$$
$$- \left\{\frac{b}{2t_0} \mp \frac{\Delta}{t_0} + O\left(\Delta^2\right)\right\} R\left(2z^\pm, 2z_0\right). \tag{S20}$$

Replacing b by $b/2$ in Lemma A5 also yields

$$P\left(2z^\pm, 2z_0\right) = \frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_0}} \left(\frac{\Delta}{b^{1/2}}\right) + O\left(\frac{\Delta^2}{b^{1/2}}\right), \text{ and} \tag{S21}$$
$$R\left(2z^\pm, 2z_0\right) = \frac{b^{1/2}}{2\sqrt{\pi}\sqrt{t_0}} \left[1 \mp \frac{\Delta}{2t_0} - \frac{1}{t_0}\left(\frac{\Delta^2}{b}\right) + O\left\{\max\left(b, \frac{\Delta^3}{b}\right)\right\}\right]. \tag{S22}$$

Substituting (S11), (S20), (S21), and (S22) into (S18) and then recognizing that the $O(\Delta)$ term in (S11) and the $O\left(\Delta^2/b^{1/2}\right)$ term in (S21) are both at most $o\left(b^{1/2}\right)$, we can obtain

$$V_{d_0}^{2\pm} = d_0 \frac{b^{-3/2}}{4\sqrt{\pi} t_0^{3/2}} \left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_0}} \left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}. \tag{S23}$$

(ii) Approximation to $V_{d_0}^{+-}$. Our next task is to find an approximation to

$$V_{d_0}^{+-} = d_0 \frac{B(t_0)}{b^2} \int_0^{t_0} \left[\left\{ \ln u - \ln b - \Psi\left(z^- + 1\right) \right\} \left\{ \ln u - \ln b - \Psi\left(z^+ + 1\right) \right\} \right.$$
$$\left. \times \frac{u^{2z_0} \exp(-2u/b)}{(b/2)^{2z_0+1} \Gamma(2z_0+1)} \right] du.$$

By Lemma A1,

$$2\ln b + \Psi\left(z^- + 1\right) + \Psi\left(z^+ + 1\right) = 2\ln t_0 + \frac{b}{t_0} - \frac{\Delta^2}{t_0^2} + o\left(\Delta^2\right),$$

and

$$\left\{ \ln b + \Psi\left(z^- + 1\right) \right\} \left\{ \ln b + \Psi\left(z^+ + 1\right) \right\}$$
$$= \ln^2 t_0 + \left(\frac{\ln t_0}{t_0} \right) b - \left(\frac{1 + \ln t_0}{t_0^2} \right) \Delta^2 + o\left(\Delta^2\right).$$

Combining these with (S19) yields

$$\left\{ \ln u - \ln b - \Psi\left(z^- + 1\right) \right\} \left\{ \ln u - \ln b - \Psi\left(z^+ + 1\right) \right\}$$
$$= \ln^2 u - \left\{ 2\ln b + \Psi\left(z^- + 1\right) + \Psi\left(z^+ + 1\right) \right\} \ln u$$
$$+ \left\{ \ln b + \Psi\left(z^- + 1\right) \right\} \left\{ \ln b + \Psi\left(z^+ + 1\right) \right\}$$
$$\sim \left\{ -\frac{b}{t_0} + \frac{\Delta^2}{t_0^2} + o\left(\Delta^2\right) \right\} \left(\frac{u - t_0}{t_0} \right)$$
$$+ \left\{ 1 + \frac{b}{2t_0} - \frac{\Delta^2}{2t_0^2} + o\left(\Delta^2\right) \right\} \left(\frac{u - t_0}{t_0} \right)^2 - \frac{\Delta^2}{t_0^2} + o\left(\Delta^2\right).$$

Notice that

$$\int_0^{t_0} u^m \frac{u^{2z_0} \exp\{-u/(b/2)\}}{(b/2)^{2z_0+1} \Gamma(2z_0+1)} du = \left(\frac{b}{2} \right)^m \frac{\gamma(2z_0 + m + 1, 2z_0)}{\Gamma(2z_0+1)}$$

for $m \geq 0$. Again by Lemma A4 and straightforward calculations, we have

$$\int_0^{t_0} \frac{u^{2z_0} \exp\{-u/(b/2)\}}{(b/2)^{2z_0+1} \Gamma(2z_0+1)} du = P(2z_0, 2z_0) - R(2z_0, 2z_0),$$

$$\int_0^{t_0} \left(\frac{u - t_0}{t_0} \right) \frac{u^{2z_0} \exp\{-u/(b/2)\}}{(b/2)^{2z_0+1} \Gamma(2z_0+1)} du = \frac{b}{2t_0} P(2z_0, 2z_0) - \left(1 + \frac{b}{2t_0} \right) R(2z_0, 2z_0),$$

and

$$\int_0^{t_0} \left(\frac{u - t_0}{t_0} \right)^2 \frac{u^{2z_0} \exp\{-u/(b/2)\}}{(b/2)^{2z_0+1} \Gamma(2z_0+1)} du$$
$$= \left(\frac{b}{2t_0} + \frac{b^2}{2t_0^2} \right) P(2z_0, 2z_0) - \left(\frac{3b}{2t_0} + \frac{b^2}{2t_0^2} \right) R(2z_0, 2z_0).$$

Hence,

$$\int_0^{t_0} \left[\left\{\ln u - \ln b - \Psi\left(z^- + 1\right)\right\}\left\{\ln u - \ln b - \Psi\left(z^+ + 1\right)\right\}\right.$$
$$\left.\times \frac{u^{2z_0}\exp\{-u/(b/2)\}}{(b/2)^{2z_0+1}\Gamma(2z_0+1)}\right] du$$
$$\sim \left\{\frac{b}{2t_0} - \frac{\Delta^2}{t_0^2} + o\left(\Delta^2\right)\right\} P\left(2z_0, 2z_0\right) - \left\{\frac{b}{2t_0} + o\left(\Delta^2\right)\right\} R\left(2z_0, 2z_0\right). \qquad \text{(S24)}$$

In addition, letting $x \downarrow 0$ in equation (1) of Pagurova (1965), we have

$$P\left(2z_0, 2z_0\right) = \frac{1}{2} + \frac{b^{1/2}}{6\sqrt{\pi}\sqrt{t_0}} + O\left(b^{3/2}\right). \qquad \text{(S25)}$$

Moreover, by (S1), $R\left(2z_0, 2z_0\right)$ can be approximated as

$$R\left(2z_0, 2z_0\right) = \frac{b^{1/2}}{2\sqrt{\pi}\sqrt{t_0}}\left\{1 + O(b)\right\}. \qquad \text{(S26)}$$

It follows from (S15), (S24), (S25), and (S26) that

$$V_{d_0}^{+-} = d_0 \frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{\frac{1}{2} - \frac{3}{2t_0}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}. \qquad \text{(S27)}$$

(iii) Proof of (S9). It follows from (S23), (S27) and $b^{1/2} = o\left(\Delta^2/b\right)$ that

$$V_{d_0} = d_0 \frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{1 + O\left(b^{1/2}\right)\right\} - d_0 \frac{b^{-3/2}}{4\sqrt{\pi}t_0^{3/2}}\left\{1 - \frac{3}{t_0}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}$$
$$\sim \left(\frac{d_0}{2}\right)\frac{3}{2\sqrt{\pi}t_0^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right).$$

which establishes (S9). This completes the proof. ■

S.4.7 Proof of Lemma A8

It holds that

$$\left|\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}(\xi) - \left(-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\right)\right|$$
$$\leq \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(\xi) - \hat{J}^{(2)}(t_0)\right| + \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(t_0) - E\left\{\hat{J}^{(2)}(t_0)\right\}\right|$$
$$+ \left|\left(\frac{b^{3/2}}{\Delta}\right)E\left\{\hat{J}^{(2)}(t_0)\right\} - \left(-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\right)\right|$$
$$= D_1 + D_2 + D_3 \ \text{(say)}.$$

Then, the proof boils down to establishing the following three statements:

$$D_1 = \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(\xi) - \hat{J}^{(2)}(t_0)\right| = o_p(1). \tag{S28}$$

$$D_2 = \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(t_0) - E\left\{\hat{J}^{(2)}(t_0)\right\}\right| = o_p(1). \tag{S29}$$

$$D_3 = \left|\left(\frac{b^{3/2}}{\Delta}\right) E\left\{\hat{J}^{(2)}(t_0)\right\} - \left(-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\right)\right| = o(1). \tag{S30}$$

We work on (S30) first and then proceed to (S28) and (S29).

S.4.7.1 Proof of (S30)

The proof is similar to that of Lemma A6, and thus we provide only its outline. Because

$$E\left\{\ddot{K}_{t_0}^{\pm}(X_i)\right\} = \int_0^\infty \ddot{K}_{t_0}^{\pm}(u)\, g(u)\, du + d_0 \int_0^{t_0} \ddot{K}_{t_0}^{\pm}(u)\, du =: I_g^{\pm} + I_{d_0}^{\pm},$$

where

$$\ddot{K}_{t_0}^{\pm}(u) = \frac{1}{b^2}\left[\left\{\ln u - \ln b - \Psi\left(z^{\pm}+1\right)\right\}^2 - \Psi^{(1)}\left(z^{\pm}+1\right)\right] K_{t_0}^{\pm}(u),$$

it holds that

$$E\left\{\hat{J}^{(2)}(t_0)\right\} = E\left\{\ddot{K}_{t_0}^{-}(X_i)\right\} - E\left\{\ddot{K}_{t_0}^{+}(X_i)\right\} = \left(I_g^{-} - I_g^{+}\right) + \left(I_{d_0}^{-} - I_{d_0}^{+}\right).$$

A similar procedure to the proof of Lemma A6 yields

$$I_g^{\pm} = g^{(2)}(t_0) \pm g^{(3)}(t_0)\,\Delta + o(\Delta)$$

so that

$$I_g^{-} - I_g^{+} = -2g^{(3)}(t_0)\,\Delta + o(\Delta).$$

It can be also found via straightforward but tedious calculations that

$$I_{d_0}^{\pm} \sim \frac{d_0}{b^2}\left[\left\{-\frac{\Delta^2}{2t_0^2} \pm \frac{3\Delta b}{2t_0^2} + o(\Delta b)\right\} P\left(z^{\pm}, z_0\right) - \left\{\mp\frac{\Delta}{t_0} + \frac{b}{t_0} + o(b)\right\} R\left(z^{\pm}, z_0\right)\right].$$

Then, using Lemma A5 yields

$$\begin{aligned} I_{d_0}^{-} - I_{d_0}^{+} &= \frac{d_0}{b^2}\left[\left\{O(\Delta b) + O\left(\frac{\Delta^3}{b^{1/2}}\right)\right\} - \frac{b^{1/2}}{\sqrt{2\pi}\sqrt{t_0}}\left\{\frac{2\Delta}{t_0} + O(\Delta b)\right\}\right] \\ &= -\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{3/2}}\right) + O\left(\frac{\Delta^3}{b^{1/2}}\right). \end{aligned}$$

Therefore,

$$E\left\{\hat{J}^{(2)}(t_0)\right\} = O(\Delta) + \left\{-\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{3/2}}\right) + O\left(\frac{\Delta^3}{b^{1/2}}\right)\right\}$$
$$= -\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{3/2}}\right) + O(\Delta)$$
$$\sim -\sqrt{\frac{2}{\pi}}\frac{d_0}{t_0^{3/2}}\left(\frac{\Delta}{b^{3/2}}\right),$$

and thus (S30) is demonstrated.

S.4.7.2 Proof of (S28)

A mean-value expansion of $\hat{J}^{(2)}(\xi)$ around $\xi = t_0$ yields

$$\hat{J}^{(2)}(\xi) - \hat{J}^{(2)}(t_0) = \hat{J}^{(3)}(\tau)(\xi - t_0)$$
$$= \hat{J}^{(3)}(t_0)(\xi - t_0) + \left\{\hat{J}^{(3)}(\tau) - \hat{J}^{(3)}(t_0)\right\}(\xi - t_0)$$

for some τ on the line segment joining ξ and t_0. It can be shown that the second term is of smaller order than the first term by Lipschitz continuity of $\hat{J}^{(3)}(\cdot)$. To put it another way, the order of magnitude in D_1 is determined by that of $\left(b^{3/2}/\Delta\right)\left|\hat{J}^{(3)}(t_0)\right||\xi - t_0|$.

In what follows, the order of magnitude in $\hat{J}^{(3)}(t_0)$ can be found via the identity $\hat{J}^{(3)}(t_0) \equiv E\left\{\hat{J}^{(3)}(t_0)\right\} + \left[\hat{J}^{(3)}(t_0) - E\left\{\hat{J}^{(3)}(t_0)\right\}\right]$. As in the proofs of Lemma A6 and (S30),

$$\left|E\left\{\hat{J}^{(3)}(t_0)\right\}\right| \le E\left|\dddot{K}_{t_0}^{-}(X_i) - \dddot{K}_{t_0}^{+}(X_i)\right|$$
$$= \int_0^\infty \left|\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u)\right| g(u)\,du + |d_0|\int_0^{t_0}\left|\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u)\right| du,$$

where

$$\dddot{K}_{t_0}^{\pm}(u) = \frac{1}{b^3}\left[\left\{\ln u - \ln b - \Psi\left(z^{\pm}+1\right)\right\}^3 \right.$$
$$\left. -3\Psi^{(1)}\left(z^{\pm}+1\right)\left\{\ln u - \ln b - \Psi\left(z^{\pm}+1\right)\right\} - \Psi^{(2)}\left(z^{\pm}+1\right)\right] K_{t_0}^{\pm}(u),$$

and $\int_0^\infty \left|\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u)\right| g(u)\,du$ is at most $O(1)$ by Lemma A1 and boundedness of $g^{(3)}(\cdot)$ on I_0. It turns out that $|d_0|\int_0^{t_0}\left|\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u)\right| du$ is the dominant term in $\left|E\left\{\hat{J}^{(3)}(t_0)\right\}\right|$. A mean-value expansion of $\dddot{K}_{t_0}^{\pm}(u)$ around $\Delta = 0$ implies that

$$\begin{aligned}\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u) \sim 2\left(\frac{\Delta}{b^4}\right)\Big[&-\{\ln u - \ln b - \Psi(z_0+1)\}^4 \\ &+ 6\Psi^{(1)}(z_0+1)\{\ln u - \ln b - \Psi(z_0+1)\}^2 \\ &+ 4\Psi^{(2)}(z_0+1)\{\ln u - \ln b - \Psi(z_0+1)\} \\ &-3\left\{\Psi^{(1)}(z_0+1)\right\}^2 + \Psi^{(3)}(z_0+1)\Big] K_{t_0}(u).\end{aligned}$$

By extending Lemma A.1 of Funke and Hirukawa (2024) to higher-order moments of the log-transformed gamma random variable, taking a similar procedure to the proof of Lemma A7 and employing Lemma A1, it can be found that

$$\int_0^{t_0}\left|\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u)\right| du \le \int_0^{\infty}\left|\dddot{K}_{t_0}^{-}(u) - \dddot{K}_{t_0}^{+}(u)\right| du = O\left(\frac{\Delta}{b^4}\right) O\left(b^2\right) = O\left(\frac{\Delta}{b^2}\right).$$

Therefore, $\left|E\left\{\hat{J}^{(3)}(t_0)\right\}\right| \le O(1) + O(\Delta/b^2) = O(\Delta/b^2)$.

Furthermore,

$$Var\left\{\hat{J}^{(3)}(t_0)\right\} = \frac{1}{n}\left[E\left\{\dddot{K}_{t_0}^{-}(X_i) - \dddot{K}_{t_0}^{+}(X_i)\right\}^2 - E^2\left\{\dddot{K}_{t_0}^{-}(X_i) - \dddot{K}_{t_0}^{+}(X_i)\right\}\right],$$

where $\left|E\left\{\dddot{K}_{t_0}^{-}(X_i) - \dddot{K}_{t_0}^{+}(X_i)\right\}\right| = \left|E\left\{\hat{J}^{(3)}(t_0)\right\}\right| = O(\Delta/b^2)$ as above. By Lemma A1 and a similar procedure to the proof of Lemma A7, it can be shown that

$$E\left\{\dddot{K}_{t_0}^{-}(X_i) - \dddot{K}_{t_0}^{+}(X_i)\right\}^2 = O\left(\frac{\Delta^2}{b^8}\right) O\left(b^{7/2}\right) = O\left(\frac{\Delta^2}{b^{9/2}}\right).$$

It follows that $Var\left\{\hat{J}^{(3)}(t_0)\right\} = O\left\{\Delta^2/\left(nb^{9/2}\right)\right\}$.

In conclusion,

$$\hat{J}^{(3)}(t_0) = O\left(\frac{\Delta}{b^2}\right) + O_p\left(\sqrt{\frac{\Delta^2}{nb^{9/2}}}\right).$$

Because ξ lies between $\hat{t}$ and t_0 and $\left|\hat{t} - t_0\right| = O_p(c_n) = o_p\left(b^{1/2}\right)$ by Theorem 1, it holds that $|\xi - t_0| \le \left|\hat{t} - t_0\right| = o_p\left(b^{1/2}\right)$. Therefore,

$$\begin{aligned}\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(3)}(t_0)\right| |\xi - t_0| &= \left(\frac{b^{3/2}}{\Delta}\right)\left\{O\left(\frac{\Delta}{b^2}\right) + O_p\left(\sqrt{\frac{\Delta^2}{nb^{9/2}}}\right)\right\} o_p\left(b^{1/2}\right) \\ &= o_p(1) + o_p\left(\frac{1}{\sqrt{nb^{1/2}}}\right) \xrightarrow{p} 0,\end{aligned}$$

which establishes (S28).

S.4.7.3 Proof of (S29)

It suffices to show that $Var\left\{\left(b^{3/2}/\Delta\right)\hat{J}^{(2)}(t_0)\right\} = o(1)$. To do so, again we focus on the order of magnitude in $E\left\{\ddot{K}_{t_0}^{-}(X_i) - \ddot{K}_{t_0}^{+}(X_i)\right\}^2$. By a mean-value expansion of $\ddot{K}_{t_0}^{\pm}(u)$ around $\Delta = 0$,

$$\begin{aligned}
&\ddot{K}_{t_0}^{-}(u) - \ddot{K}_{t_0}^{+}(u) \\
&\sim 2\left(\frac{\Delta}{b^3}\right)\left[-\left\{\ln u - \ln b - \Psi(z_0+1)\right\}^3\right. \\
&\left.+3\left\{\ln u - \ln b - \Psi(z_0+1)\right\}\Psi^{(1)}(z_0+1) + \Psi^{(2)}(z_0+1)\right] K_{t_0}(u).
\end{aligned}$$

Then, by a similar procedure to the proof of (S28) above,

$$E\left\{\ddot{K}_{t_0}^{-}(X_i) - \ddot{K}_{t_0}^{+}(X_i)\right\}^2 = O\left(\frac{\Delta^2}{b^6}\right)O\left(b^{5/2}\right) = O\left(\frac{\Delta^2}{b^{7/2}}\right).$$

It follows that

$$Var\left\{\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}(t_0)\right\} = \left(\frac{b^3}{\Delta^2}\right)O\left(\frac{\Delta^2}{nb^{7/2}}\right) = O\left(\frac{1}{nb^{1/2}}\right) \to 0.$$

This completes the proof. ■

S.4.8 Proof of Lemma A9

Basically, a similar strategy to the proofs of (S28) and (S29) in Lemma A8 may be taken. The derivation is straightforward but much more tedious, and thus details are omitted. ■

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