A Detailed Proof of Lemma 2 in "A Two-Stage Plug-In Bandwidth Selection and Its Implementation for Covariance Estimation"

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First Draft: May 2006
This Version: May 2007

The proof closely follows that of Theorem 9 in Chapter V of Hannan (1970). The result in Hannan (1970, p.313) gives

$$\begin{aligned} & TCov\left(\tilde{\Gamma}_h(i),\tilde{\Gamma}_h(j)\right) \\ = & \sum_{u=-\infty}^{\infty}\left\{\Gamma_h(u)\Gamma_h(u+i-j)+\Gamma_h(u+i)\Gamma_h(u-j)+\kappa_h(i,u,u+j)\right\}\varphi_T(u,i,j), \end{aligned} \tag{1}$$

where $\kappa_h(\cdot,\cdot,\cdot)$ is the fourth-order cumulant generated by the process $\{h_t\}$, and $\varphi_T(u,i,j)$ is defined for $(T-1\geq)\, i\geq j\,(\geq 1)$ by[1]

$$\varphi_T(u,i,j)=\begin{cases} 0 & \text{if } u\leq -T+i \\ 1-(i-u)/T & \text{if } -T+i\leq u\leq 0 \\ 1-i/T & \text{if } 0\leq u\leq i-j \\ 1-(j+u)/T & \text{if } i-j\leq u\leq T-j \\ 0 & \text{if } T-j\leq u \end{cases}.$$

It is easy to see that $0\leq\varphi_T(u,i,j)\leq 1$. Using (1) yields

$$\begin{aligned} \frac{T}{b_T^{2q+1}}Var(\tilde{s}^{(q)}) &= \frac{1}{b_T}\sum_{i=-(T-1)}^{T-1}\sum_{j=-(T-1)}^{T-1}\left|\frac{i}{b_T}\right|^q\left|\frac{j}{b_T}\right|^q l(\frac{i}{b_T})l(\frac{j}{b_T})\sum_{u=-\infty}^{\infty}\Gamma_h(u)\Gamma_h(u+i-j)\varphi_T(u,i,j) \\ &\quad +\frac{1}{b_T}\sum_{i=-(T-1)}^{T-1}\sum_{j=-(T-1)}^{T-1}\left|\frac{i}{b_T}\right|^q\left|\frac{j}{b_T}\right|^q l(\frac{i}{b_T})l(\frac{j}{b_T})\sum_{u=-\infty}^{\infty}\Gamma_h(u+i)\Gamma_h(u-j)\varphi_T(u,i,j) \\ &\quad +\frac{1}{b_T}\sum_{i=-(T-1)}^{T-1}\sum_{j=-(T-1)}^{T-1}\left|\frac{i}{b_T}\right|^q\left|\frac{j}{b_T}\right|^q l(\frac{i}{b_T})l(\frac{j}{b_T})\sum_{u=-\infty}^{\infty}\kappa_h(i,u,u+j)\varphi_T(u,i,j) \\ &\equiv V_1+V_2+V_3. \end{aligned}$$

Let $v\equiv i-j$. Then, V_1 can be rewritten as

$$V_1=\sum_{v=-2(T-1)}^{2(T-1)}\sum_{u=-\infty}^{\infty}\Gamma_h(u)\Gamma_h(u+v)\left\{\frac{1}{b_T}\sum_j\varphi_T(u,j+v,j)\left|\frac{j}{b_T}\right|^q l(\frac{j}{b_T})\left|\frac{j+v}{b_T}\right|^q l(\frac{j+v}{b_T})\right\},$$

[1] Anderson (1971, p.527-528, Problem 19 on p.555) defines $T\phi_T(r;g,h)$ (an equivalent to $\varphi_T(u,i,j)$) for a general triplet (r,g,h).

where the summation over j runs only for $\{j : |j| \leq T-1, |j+v| \leq T-1\}$. Pick a trimming function $m_T = O\left(b_T^{1-\epsilon}\right)$ for some $\epsilon \in (0,1)$. Then,

$$V_1 \sim \sum_{|u| \leq m_T} \sum_{|u+v| \leq m_T} \Gamma_h(u)\Gamma_h(u+v) \left\{ \frac{1}{b_T} \sum_j \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}) \right\}, \quad (2)$$

because the object inside the bracket is bounded (as shown below) and $\sum_{u=-\infty}^{\infty} |\Gamma_h(u)| < \infty \Longrightarrow \sum_{|u| \geq m_T+1}^{\infty} |\Gamma_h(u)| \to 0$ as $T \to \infty$.

Next, pick another trimming function $M_T = O\left(b_T^{1+\eta}\right)$ for some $\eta \in (0, \epsilon/(2q+1))$. Then, by $|\varphi_T(\cdot,\cdot,\cdot)| \leq 1$ and A1(c),

$$\begin{aligned} & \frac{1}{b_T} \left| \sum_{|j| \geq M_T+1}^{T-1} \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}) \right| \\ \leq \; & \frac{c}{b_T} \sum_{|j| \geq M_T+1}^{T-1} \left| \frac{j}{b_T} \right|^q \left| \frac{j}{b_T} \right|^{-b_1} \left| \frac{j+v}{b_T} \right|^q \left| \frac{j+v}{b_T} \right|^{-b_1} \\ = \; & c b_T^{2(b_1-q)-1} \sum_{|j| \geq M_T+1}^{T-1} |j|^{q-b_1} |j+v|^{q-b_1}. \end{aligned} \quad (3)$$

For an arbitrarily large T, we have $0 \leq |v| \leq 2m_T < M_T + 1 \leq |j| \leq T-1$. It follows from $b_1 - q > 1$ that $|j+v|^{q-b_1} \leq |M_T + 1 - 2m_T|^{q-b_1} = O\left(M_T^{q-b_1}\right)$. By $b_1 - q > 1$, we also have $\sum_{|j| \geq M_T+1}^{T-1} |j|^{q-b_1} = O\left(M_T^{q-b_1+1}\right)$ and $2(b_1 - q) - 1 > 1$. Hence, the right-hand side of (3) is bounded by $O\left((b_T/M_T)^{2(b_1-q)-1}\right) = O\left(b_T^{-\eta(2(b_1-q)-1)}\right) = o(1)$.[2] Therefore,

$$\begin{aligned} & \frac{1}{b_T} \sum_j \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}) \\ \sim \; & \frac{1}{b_T} \sum_{|j| \leq M_T} \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}). \end{aligned} \quad (4)$$

Furthermore, consider

$$\begin{aligned} & \frac{1}{b_T} \sum_{|j| \leq M_T} \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}) - \frac{1}{b_T} \sum_{|j| \leq M_T} \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^{2q} l^2(\frac{j}{b_T}) \\ = \; & \frac{1}{b_T} \sum_{|j| \leq M_T} \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left\{ \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}) - \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \right\}. \end{aligned} \quad (5)$$

[2] If the kernel $l(x)$ has a bounded support, i.e. $l(x) = 0$ for $|x| > 1$, then $b_T < M_T + 1 \leq |j|$ ($\Rightarrow |j/b_T| > 1$) for an arbitrarily large T and thus

$$\frac{1}{b_T} \sum_{|j| \geq M_T+1}^{T-1} \varphi_T(u, j+v, j) \left| \frac{j}{b_T} \right|^q l(\frac{j}{b_T}) \left| \frac{j+v}{b_T} \right|^q l(\frac{j+v}{b_T}) = 0.$$

Observe that

$$\begin{aligned}\left|\left|\frac{j+v}{b_T}\right|^q-\left|\frac{j}{b_T}\right|^q\right| &= \left|\left(\left|\frac{j+v}{b_T}\right|-\left|\frac{j}{b_T}\right|\right)\left(\left|\frac{j+v}{b_T}\right|^{q-1}+\left|\frac{j+v}{b_T}\right|^{q-2}\left|\frac{j}{b_T}\right|+\cdots+\left|\frac{j}{b_T}\right|^{q-1}\right)\right| \\ &\leq \left|\frac{v}{b_T}\right|\left(\left|\frac{j+v}{b_T}\right|^{q-1}+\left|\frac{j+v}{b_T}\right|^{q-2}\left|\frac{j}{b_T}\right|+\cdots+\left|\frac{j}{b_T}\right|^{q-1}\right) \\ &= O\left(\frac{m_T}{b_T}\left(\frac{M_T}{b_T}\right)^{q-1}\right),\end{aligned}$$

and

$$\left|l(\frac{j+v}{b_T})-l(\frac{j}{b_T})\right|\leq c\left|\frac{v}{b_T}\right|=O\left(\frac{m_T}{b_T}\right)$$

by A1(b). Then, by $|\varphi_T(\cdot,\cdot,\cdot)|\leq 1$, $|l(\cdot)|\leq 1$ and $\sum_{|j|\leq M_T}|j|^q=O\left(M_T^{q+1}\right)$, the right-hand side of (5) is bounded by

$$\begin{aligned}&\frac{1}{b_T}\sum_{|j|\leq M_T}\left|\frac{j}{b_T}\right|^q\left|l(\frac{j}{b_T})\right|\left|\left|\frac{j+v}{b_T}\right|^q l(\frac{j+v}{b_T})-\left|\frac{j}{b_T}\right|^q l(\frac{j}{b_T})\right| \\ \leq\ &\frac{1}{b_T}\sum_{|j|\leq M_T}\left|\frac{j}{b_T}\right|^q\left|\left(\left|\frac{j}{b_T}\right|^q+O\left(\frac{m_T}{b_T}\left(\frac{M_T}{b_T}\right)^{q-1}\right)\right)\left(l(\frac{j}{b_T})+O\left(\frac{m_T}{b_T}\right)\right)-\left|\frac{j}{b_T}\right|^q l(\frac{j}{b_T})\right| \\ \leq\ &\frac{1}{b_T}\sum_{|j|\leq M_T}\left|\frac{j}{b_T}\right|^q O\left(\frac{m_T}{b_T}\left(\frac{M_T}{b_T}\right)^{q-1}\right)+\frac{1}{b_T}\sum_{|j|\leq M_T}\left|\frac{j}{b_T}\right|^{2q}O\left(\frac{m_T}{b_T}\right) \\ &+\frac{1}{b_T}\sum_{|j|\leq M_T}\left|\frac{j}{b_T}\right|^q O\left(\left(\frac{m_T}{b_T}\right)^2\left(\frac{M_T}{b_T}\right)^{q-1}\right) \\ =\ &O\left(\frac{m_T}{b_T}\left(\frac{M_T}{b_T}\right)^{2q}\right)+O\left(\frac{m_T}{b_T}\left(\frac{M_T}{b_T}\right)^{2q+1}\right)+O\left(\left(\frac{m_T}{b_T}\right)^2\left(\frac{M_T}{b_T}\right)^{2q}\right) \\ =\ &O\left(\frac{m_T}{b_T}\left(\frac{M_T}{b_T}\right)^{2q+1}\right) \\ =\ &O\left(b_T^{-\epsilon+\eta(2q+1)}\right)=o(1).\end{aligned}$$

Therefore,

$$\frac{1}{b_T}\sum_{|j|\leq M_T}\varphi_T(u,j+v,j)\left|\frac{j}{b_T}\right|^q l(\frac{j}{b_T})\left|\frac{j+v}{b_T}\right|^q l(\frac{j+v}{b_T})\sim\frac{1}{b_T}\sum_{|j|\leq M_T}\varphi_T(u,j+v,j)\left|\frac{j}{b_T}\right|^{2q}l^2(\frac{j}{b_T}). \quad (6)$$

Finally, $q>\left(-1+\sqrt{5}\right)/4$ by A2(a), which implies that $\eta<\epsilon/(2q+1)<1/(2q+1)<2q \Rightarrow M_T/T\to 0$ by A3(a). For (u,v,j) such that $|u|\leq m_T$, $|u+v|\leq m_T$, and $|j|\leq M_T$, the domain over which $\varphi_T(u,j+v,j)=0$ vanishes as $T\to\infty$. The support of $\varphi_T(u,j+v,j)$ approaches to the entire real line, and over this support,

$$|\varphi_T(u,j+v,j)-1|\leq\frac{|u|+|j+v|+|j|}{T}\leq\frac{m_T+(M_T+2m_T)+M_T}{T}\to 0. \quad (7)$$

Using (7) and A1(a) gives

$$\frac{1}{b_T}\sum_{|j|\leq M_T}\varphi_T(u,j+v,j)\left|\frac{j}{b_T}\right|^{2q}l^2(\frac{j}{b_T})\sim\frac{1}{b_T}\sum_{|j|\leq M_T}\left|\frac{j}{b_T}\right|^{2q}l^2(\frac{j}{b_T})\to\int_{-\infty}^{\infty}|x|^{2q}l^2(x)\,dx<\infty. \quad (8)$$

We also have

$$\sum_{|u|\leq m_T} \Gamma_h(u) \to \sum_{u=-\infty}^{\infty} \Gamma_h(u) = s^{(0)} < \infty. \tag{9}$$

Combining (2) with (4)(6)(8)(9) finally yields

$$V_1 \sim \left\{ \sum_{|u|\leq m_T} \Gamma_h(u) \right\}^2 \left\{ \frac{1}{b_T} \sum_{|j|\leq M_T} \left| \frac{j}{b_T} \right|^{2q} l^2(\frac{j}{b_T}) \right\} \to \left(s^{(0)} \right)^2 \int_{-\infty}^{\infty} |x|^{2q} \, l^2 \, (x) \, dx < \infty.$$

Similarly, we have $V_2 \to \left(s^{(0)} \right)^2 \int_{-\infty}^{\infty} |x|^{2q} \, l^2 \, (x) \, dx$. Lastly, by A1(a) and A4(c),

$$|V_3| \leq \frac{1}{b_T} \left(\sup_{x\geq 0} |x|^q \, |l(x)| \right)^2 \sum_{i=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} |\kappa_h(i,u,v)| \to 0,$$

which establishes the first approximation. The second approximation is a standard result of spectral density estimation. The third approximation can be shown by recognizing that $\int_{-\infty}^{\infty} |x|^q \, l^2(x) dx < \infty$ by A1(a). ■

References

[1] Anderson, T. W. (1971): *The Statistical Analysis of Time Series*. New York: John Wiley & Sons.

[2] Hannan, E. J. (1970): *Multiple Time Series*. New York: John Wiley & Sons.