Supplement to "Nonparametric Estimation of Splicing Points in Actuarial Loss Distributions via Data Transformation"

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S.1 Introduction

In the proofs of lemmata below, equation numbers with no letters and starting from the letter "A" correspond to those given in the main body and in the Appendix, respectively, unless otherwise noted. In addition to the shorthand notation given at the beginning of the Appendix, we newly introduce the beta random variable $Z_{b,y,m}^{0} \overset{d}{=} Beta\left(p_{b,y,m}^{0}, q_{b,y}^{0}\right) := Beta\left\{y/b+m+1, (1-y)/b+1\right\}$ for $m \in \{0,1,2,\ldots\}$. Finally, notice that it is easiest and fastest to verify all the calculations with the aid of Maple™ or Mathematica®.

S.1.1 List of Useful Formulae

Before proceeding, we present the formulae that are useful for the proofs below. While (S2) is taken from Lemma A.2 of Funke and Hirukawa (2024), (S3) is a natural extension of the lemma and can be shown immediately. Moreover, (S4) is the same as equation (A3) of Funke and Hirukawa (2025), and (S5)-(S7) are taken from Lemma A1 of Funke and Hirukawa (2025) with "$x \in I_0$" replaced by "$y \in I_T$".

S.1.1.1 Stirling's Formula

As $a \to \infty$,

$$\Gamma(a+1) = \sqrt{2\pi} a^{a+1/2} \exp(-a) \left\{1 + \frac{1}{12a} + O\left(a^{-2}\right)\right\}. \tag{S1}$$

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S.1.1.2 Moments of the Log-Transformed Beta Random Variable

Let $Z \overset{d}{=} Beta\left(\gamma, \delta\right)$. Then, for $m \in \{0, 1, 2, ...\}$,

$$E\left\{Z^m \ln\left(\frac{Z}{1-Z}\right)\right\} = \frac{B\left(\gamma+m,\delta\right)}{B\left(\gamma,\delta\right)}\left\{\Psi\left(\gamma+m\right) - \Psi\left(\delta\right)\right\}, \tag{S2}$$

and

$$E\left\{Z^m \ln^2\left(\frac{Z}{1-Z}\right)\right\} \\ = \frac{B\left(\gamma+m,\delta\right)}{B\left(\gamma,\delta\right)}\left[\left\{\Psi\left(\gamma+m\right) - \Psi\left(\delta\right)\right\}^2 + \Psi^{(1)}\left(\gamma+m\right) + \Psi^{(1)}\left(\delta\right)\right], \tag{S3}$$

where

$$\frac{B\left(\gamma+m,\delta\right)}{B\left(\gamma,\delta\right)} = \begin{cases} 1 & \text{for } m = 0 \\ \dfrac{\prod_{k=1}^{m}(\gamma+k-1)}{\prod_{k=1}^{m}(\gamma+\delta+k-1)} & \text{for } m \geq 1 \end{cases}.$$

S.1.1.3 Recursive Formulae for Digamma and Trigamma Functions

For $a > 0$ and $m \in \{0, 1, 2, ...\}$,

$$\Psi^{(m)}\left(a+1\right) = \Psi^{(m)}\left(a\right) + \frac{(-1)^m m!}{a^{m+1}}. \tag{S4}$$

S.1.1.4 Approximations to Digamma and Trigamma Functions

As $b \to 0$,

$$\sup_{y\in I_T}\left|\Psi\left(\frac{y}{b}+1\right) - \left\{\ln\left(\frac{y}{b}\right) + \frac{b}{2y}\right\}\right| = O\left(b^2\right), \tag{S5}$$

$$\sup_{y\in I_T}\left|\Psi^{(1)}\left(\frac{y}{b}+1\right) - \left(\frac{b}{y} - \frac{b^2}{2y^2}\right)\right| = O\left(b^3\right), \tag{S6}$$

and for $m \geq 2$,

$$\sup_{y\in I_T}\left|\Psi^{(m)}\left(\frac{y}{b}+1\right)\right| = O\left(b^m\right). \tag{S7}$$

S.2 Proofs of Lemmata

S.2.1 Proof of Lemma A1

The proof closely follows the one of Lemma A.1 in Moscovich, Nadler and Spiegelman (2016). The beta random variable $Z \overset{d}{=} Beta\left(p, q\right)$ has the density

$$f_Z\left(z\right) = \frac{z^{p-1}\left(1-z\right)^{q-1}}{B\left(p,q\right)} = \frac{\Gamma\left(p+q\right)}{\Gamma\left(p\right)\Gamma\left(q\right)}z^{p-1}\left(1-z\right)^{q-1}.$$

By the change of variables $z := \mu + \sigma v$ with

$$(\mu, \sigma^2) = \left(\frac{p}{p+q}, \frac{pq}{(p+q)^2 (p+q+1)} \right),$$

the density can be rearranged to

$$\begin{aligned} f_Z(z) &= f_Z(\mu + \sigma v) \\ &= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} (\mu + \sigma v)^{p-1} (1 - \mu - \sigma v)^{q-1} \\ &= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \mu^{p-1} (1-\mu)^{q-1} \left(1 + \frac{\sigma}{\mu} v\right)^{p-1} \left(1 - \frac{\sigma}{1-\mu} v\right)^{q-1}. \end{aligned}$$

Then, the proof boils down to demonstrating that

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \mu^{p-1} (1-\mu)^{q-1} = \frac{1}{\sqrt{2\pi}\sigma} \left\{1 + O\left(p^{-1}\right)\right\}, \tag{S8}$$

and that

$$\begin{aligned} &\left(1 + \frac{\sigma}{\mu} v\right)^{p-1} \left(1 - \frac{\sigma}{1-\mu} v\right)^{q-1} \\ &= \exp\left(-\frac{v^2}{2}\right) \left[1 + \frac{v}{\sqrt{p+q+1}} \left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right) \right. \\ &\left. + \frac{v^3}{3} \left\{ \left(\frac{q}{p+q}\right)^{3/2} \frac{1}{\sqrt{p}} - \left(\frac{p}{p+q}\right)^{3/2} \frac{1}{\sqrt{q}} \right\} + O\left(p^{-1}\right) \right], \end{aligned} \tag{S9}$$

as $p, q \to \infty$ and $p \asymp q$.

Proof of (S8). By the definition of μ,

$$\mu^{p-1} (1-\mu)^{q-1} = \frac{p^{p-1} q^{q-1}}{(p+q)^{p+q-2}}.$$

In addition, by the property of the gamma function,

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} = \left(\frac{pq}{p+q}\right) \frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)}.$$

Applying (S1) to three gamma functions under $p, q \to \infty$ and $p \asymp q$ gives

$$\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} = \frac{(p+q)^{p+q+1/2}}{\sqrt{2\pi} p^{p+1/2} q^{q+1/2}} \left\{1 + O\left(p^{-1}\right)\right\}.$$

Therefore,

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \mu^{p-1} (1-\mu)^{q-1} = \frac{(p+q)^{3/2}}{\sqrt{2\pi}\sqrt{pq}} \left\{1 + O\left(p^{-1}\right)\right\}.$$

Observe that the right-hand side can be rewritten as

$$\frac{(p+q)\sqrt{(p+q+1)-1}}{\sqrt{2\pi}\sqrt{pq}}\left\{1+O\left(p^{-1}\right)\right\}$$
$$=\frac{(p+q)\sqrt{p+q+1}}{\sqrt{2\pi}\sqrt{pq}}\sqrt{1-\frac{1}{p+q+1}}\left\{1+O\left(p^{-1}\right)\right\}.$$

Then, (S8) can be demonstrated by recognizing that

$$\frac{1}{\sigma}=\frac{(p+q)\sqrt{p+q+1}}{\sqrt{pq}}$$

and that

$$\sqrt{1-\frac{1}{p+q+1}}=1+O\left(p^{-1}\right).$$

Proof of (S9). For

$$\left(1+\frac{\sigma}{\mu}v\right)^{p-1}\left(1-\frac{\sigma}{1-\mu}v\right)^{q-1}$$
$$=\exp\left\{(p-1)\ln\left(1+\frac{\sigma}{\mu}v\right)+(q-1)\ln\left(1-\frac{\sigma}{1-\mu}v\right)\right\},$$

observe that

$$\left(\frac{\sigma}{\mu},\frac{\sigma}{1-\mu}\right)=\left(\sqrt{\frac{q}{p\left(p+q+1\right)}},\sqrt{\frac{p}{q\left(p+q+1\right)}}\right)$$

are both $O\left(p^{-1/2}\right)\to 0$. Then, by a fourth-order Taylor expansion,

$$\ln\left(1+\frac{\sigma}{\mu}v\right)$$
$$=\left(\frac{\sigma}{\mu}\right)v-\frac{1}{2}\left(\frac{\sigma}{\mu}\right)^2v^2+\frac{1}{3}\left(\frac{\sigma}{\mu}\right)^3v^3+O\left(p^{-2}\right),$$

and

$$\ln\left(1-\frac{\sigma}{1-\mu}v\right)$$
$$=-\left(\frac{\sigma}{1-\mu}\right)v-\frac{1}{2}\left(\frac{\sigma}{1-\mu}\right)^2v^2-\frac{1}{3}\left(\frac{\sigma}{1-\mu}\right)^3v^3+O\left(p^{-2}\right).$$

Hence,

$$(p-1)\ln\left(1+\frac{\sigma}{\mu}v\right)+(q-1)\ln\left(1-\frac{\sigma}{1-\mu}v\right)$$
$$=\sigma\left(\frac{p-1}{\mu}-\frac{q-1}{1-\mu}\right)v-\frac{\sigma^2}{2}\left\{\frac{p-1}{\mu^2}+\frac{q-1}{(1-\mu)^2}\right\}v^2$$
$$+\frac{\sigma^3}{3}\left\{\frac{p-1}{\mu^3}-\frac{q-1}{(1-\mu)^3}\right\}v^3+O\left(p^{-1}\right). \qquad \text{(S10)}$$

Again by the definitions of (μ, σ^2), $p, q \to \infty$ and $p \asymp q$,

$$\sigma \left(\frac{p-1}{\mu} - \frac{q-1}{1-\mu} \right) = \frac{1}{\sqrt{p+q+1}} \left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}} \right), \tag{S11}$$

and

$$\begin{aligned} \sigma^2 \left\{ \frac{p-1}{\mu^2} + \frac{q-1}{(1-\mu)^2} \right\} &= \frac{pq}{p+q+1} \left(\frac{p-1}{p^2} + \frac{q-1}{q^2} \right) \\ &= \frac{pq}{p+q+1} \left(\frac{1}{p} + \frac{1}{q} \right) \left\{ 1 + O\left(p^{-1}\right) \right\} \\ &= \frac{p+q}{p+q+1} \left\{ 1 + O\left(p^{-1}\right) \right\} \\ &= 1 + O\left(p^{-1}\right). \end{aligned} \tag{S12}$$

Moreover,

$$\begin{aligned} &\sigma^3 \left\{ \frac{p-1}{\mu^3} - \frac{q-1}{(1-\mu)^3} \right\} \\ &= \left(\frac{q}{p+q+1} \right)^{3/2} \left(\frac{p-1}{p^{3/2}} \right) - \left(\frac{p}{p+q+1} \right)^{3/2} \left(\frac{q-1}{q^{3/2}} \right), \end{aligned}$$

where

$$\begin{aligned} \frac{p}{p+q+1} &= \frac{p}{p+q} \left\{ 1 + O\left(p^{-1}\right) \right\}, \\ \frac{p-1}{p^{3/2}} &= \frac{1}{\sqrt{p}} \left\{ 1 + O\left(p^{-1}\right) \right\}, \end{aligned}$$

and so forth. It follows that

$$\begin{aligned} &\sigma^3 \left\{ \frac{p-1}{\mu^3} - \frac{q-1}{(1-\mu)^3} \right\} \\ &= \left\{ \left(\frac{q}{p+q} \right)^{3/2} \frac{1}{\sqrt{p}} - \left(\frac{p}{p+q} \right)^{3/2} \frac{1}{\sqrt{q}} \right\} \left\{ 1 + O\left(p^{-1}\right) \right\}. \end{aligned} \tag{S13}$$

Therefore, by (S10)-(S13),

$$\begin{aligned} &\left(1 + \frac{\sigma}{\mu} v \right)^{p-1} \left(1 - \frac{\sigma}{1-\mu} v \right)^{q-1} \\ &= \exp\left\{ \sigma \left(\frac{p-1}{\mu} - \frac{q-1}{1-\mu} \right) v \right\} \exp\left\{ -\frac{\sigma^2}{2} \left\{ \frac{p-1}{\mu^2} + \frac{q-1}{(1-\mu)^2} \right\} v^2 \right\} \\ &\times \exp\left\{ \frac{\sigma^3}{3} \left\{ \frac{p-1}{\mu^3} - \frac{q-1}{(1-\mu)^3} \right\} v^3 \right\} \exp\left\{ O\left(p^{-1}\right) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\underbrace{\frac{1}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)v}_{=O\left(p^{-1/2}\right)}\right\}\underbrace{\exp\left[-\frac{v^2}{2}\left\{1+O\left(p^{-1}\right)\right\}\right]}_{=\exp\left(-\frac{v^2}{2}\right)\exp\{O(p^{-1})\}} \\
&\times \exp\left[\frac{1}{3}\left\{\underbrace{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}}_{=O\left(p^{-1/2}\right)}\right\}\left\{1+O\left(p^{-1}\right)\right\}v^3\right]\underbrace{\exp\left\{O\left(p^{-1}\right)\right\}}_{=1+O(p^{-1})} \\
&= \exp\left(-\frac{v^2}{2}\right)\exp\left[\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right. \\
&\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right]\left\{1+O\left(p^{-1}\right)\right\} \\
&= \exp\left(-\frac{v^2}{2}\right)\left[1+\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right. \\
&\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right]\left\{1+O\left(p^{-1}\right)\right\} \\
&= \exp\left(-\frac{v^2}{2}\right)\left[1+\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right. \\
&\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right],
\end{aligned}
$$

and thus (S9) is established. This completes the proof. ■

S.2.2 Proof of Lemma A2

Because (A1) is a direct outcome from Lemma A1, it suffices to derive approximations to $\xi^{\pm}_{b,y,m}$ and $\zeta^{\pm}_{b,y,m}$. These are built on the following results as $n \to \infty$, where the $O(b)$ rate in each approximation is uniform on $y \in I_T$:

$$
\sqrt{\frac{p^{\pm}_{b,y,m}}{q^{\pm}_{b,y}}} = \sqrt{\frac{y}{1-y}}\left\{1 \pm \frac{\Delta}{2y(1-y)} + O(b)\right\}. \tag{S14}
$$

$$
\sqrt{\frac{q^{\pm}_{b,y}}{p^{\pm}_{b,y,m}}} = \sqrt{\frac{1-y}{y}}\left\{1 \mp \frac{\Delta}{2y(1-y)} + O(b)\right\}. \tag{S15}
$$

$$
\frac{1}{\sqrt{p^{\pm}_{b,y,m}+q^{\pm}_{b,y}+1}} = b^{1/2}\left\{1+O(b)\right\}. \tag{S16}
$$

$$
\left(\frac{p^{\pm}_{b,y,m}}{p^{\pm}_{b,y,m}+q^{\pm}_{b,y}}\right)^{3/2} = \sqrt{y}\left\{y \pm \frac{3}{2}\Delta + O(b)\right\}. \tag{S17}
$$

$$\left(\frac{q_{b,y}^{\pm}}{p_{b,y,m}^{\pm}+q_{b,y}^{\pm}}\right)^{3/2} = \sqrt{1-y}\left\{(1-y)\mp\frac{3}{2}\Delta+O(b)\right\}. \tag{S18}$$

$$\frac{1}{\sqrt{p_{b,y,m}^{\pm}}} = \frac{b^{1/2}}{\sqrt{y}}\left\{1\mp\frac{\Delta}{2y}+O(b)\right\}. \tag{S19}$$

$$\frac{1}{\sqrt{q_{b,y}^{\pm}}} = \frac{b^{1/2}}{\sqrt{1-y}}\left\{1\pm\frac{\Delta}{2(1-y)}+O(b)\right\}. \tag{S20}$$

These approximations can be obtained in a similar manner, and thus we only present the derivation of (S14). By a second-order Taylor expansion of $\sqrt{p_{b,y,m}^{\pm}/q_{b,y}^{\pm}}$ around $\Delta = 0$ and $\Delta^2 = o(b)$,

$$\begin{aligned}
\sqrt{\frac{p_{b,y,m}^{\pm}}{q_{b,y}^{\pm}}} &= \sqrt{\frac{y+(m+1)b}{1-y+b}} \pm \frac{1+(m+2)b}{2(1-y+b)^{3/2}\sqrt{y+(m+1)b}}\Delta + O\left(\Delta^2\right) \\
&= \sqrt{\frac{y}{1-y}}\{1+O(b)\} \pm \frac{1}{2(1-y)^{3/2}\sqrt{y}}\{1+O(b)\}\Delta + o(b) \\
&= \sqrt{\frac{y}{1-y}}\left\{1\pm\frac{\Delta}{2y(1-y)}+O(b)\right\}.
\end{aligned}$$

Substituting (S14)-(S20) into the definition of $\xi_{b,y,m}^{\pm}$ yields

$$\begin{aligned}
&\xi_{b,y,m}^{\pm} \\
&= \frac{b^{-1/2}}{\sqrt{p_{b,y,m}^{\pm}+q_{b,y}^{\pm}+1}}\left(\sqrt{\frac{p_{b,y,m}^{\pm}}{q_{b,y}^{\pm}}}-\sqrt{\frac{q_{b,y}^{\pm}}{p_{b,y,m}^{\pm}}}\right) \\
&= b^{-1/2}b^{1/2}\{1+O(b)\}\left[\sqrt{\frac{y}{1-y}}\left\{1\pm\frac{\Delta}{2y(1-y)}+O(b)\right\}\right. \\
&\left.-\sqrt{\frac{1-y}{y}}\left\{1\mp\frac{\Delta}{2y(1-y)}+O(b)\right\}\right] \\
&= \frac{\{1+O(b)\}}{\sqrt{y(1-y)}}\left[y\left\{1\pm\frac{\Delta}{2y(1-y)}+O(b)\right\}-(1-y)\left\{1\mp\frac{\Delta}{2y(1-y)}+O(b)\right\}\right] \\
&= \frac{2y-1}{\sqrt{y(1-y)}} \pm \frac{\Delta}{2\{y(1-y)\}^{3/2}} + O(b).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\zeta_{b,y,m}^{\pm} \\
&= \frac{b^{-1/2}}{3}\left\{\left(\frac{q_{b,y}^{\pm}}{p_{b,y,m}^{\pm}+q_{b,y}^{\pm}}\right)^{3/2}\frac{1}{\sqrt{p_{b,y,m}^{\pm}}}-\left(\frac{p_{b,y,m}^{\pm}}{p_{b,y,m}^{\pm}+q_{b,y}^{\pm}}\right)^{3/2}\frac{1}{\sqrt{q_{b,y}^{\pm}}}\right\} \\
&= \frac{b^{-1/2}}{3}\left[\sqrt{1-y}\left\{(1-y)\mp\frac{3}{2}\Delta+O(b)\right\}\frac{b^{1/2}}{\sqrt{y}}\left\{1\mp\frac{\Delta}{2y}+O(b)\right\}\right. \\
&\left.-\sqrt{y}\left\{y\pm\frac{3}{2}\Delta+O(b)\right\}\frac{b^{1/2}}{\sqrt{1-y}}\left\{1\pm\frac{\Delta}{2(1-y)}+O(b)\right\}\right] \\
&= \frac{1}{3\sqrt{y(1-y)}}\left[(1-y)\left\{(1-y)\mp\frac{1+2y}{2y}\Delta+O(b)\right\}-y\left\{y\pm\frac{3-2y}{2(1-y)}\Delta+O(b)\right\}\right] \\
&= \frac{1-2y}{3\sqrt{y(1-y)}}\mp\frac{\Delta}{6\{y(1-y)\}^{3/2}}+O(b).
\end{aligned}
$$

The proof is completed by observing that the $O(b)$ rate in each approximation is uniform on $y \in I_T$. ■

S.2.3 Proof of Lemma A4

Proof of (A2). In the proof of Proposition 2(i), it has been already demonstrated that $\Phi\left(B_{b,y,m}^{\pm}\right) \to 0$ at an exponential rate. Therefore, to approximate $C_{b,t_T,m}^{\pm}(0) = \Phi\left(A_{b,t_T,m}^{\pm}\right) - \Phi\left(B_{b,t_T,m}^{\pm}\right)$, we may safely concentrate on $\Phi\left(A_{b,t_T,m}^{\pm}\right)$, where

$$A_{b,t_T,m}^{\pm} = \frac{[\mp\Delta + \{(m+2)t_T - (m+1)\}b]\sqrt{1+(m+3)b}}{\sqrt{b\{(t_T\pm\Delta)+(m+1)b\}\{1-(t_T\pm\Delta)+b\}}}.$$

By a second-order Taylor expansion of $\Phi\left(A_{b,t_T,m}^{\pm}\right)$ around $\Delta = 0$,

$$\Phi\left(A_{b,t_T,m}^{\pm}\right) = \Phi\left(A_{b,t_T,m}^{\pm}\right)\Big|_{\Delta=0} + \phi\left(A_{b,t_T,m}^{\pm}\right)\frac{\partial A_{b,t_T,m}^{\pm}}{\partial\Delta}\Big|_{\Delta=0}\Delta + O\left(\frac{\Delta^2}{b}\right).$$

Substituting

$$A_{b,t_T,m}^{\pm}\Big|_{\Delta=0} = b^{1/2}\left\{\frac{(m+2)t_T-(m+1)}{\sqrt{t_T(1-t_T)}}\right\}\{1+O(b)\} \tag{S21}$$

into $\Phi\left(A_{b,t_T,m}^{\pm}\right)\Big|_{\Delta=0}$ and $\phi\left(A_{b,t_T,m}^{\pm}\right)\Big|_{\Delta=0}$ and then using $\Phi(0) = 1/2$, $\phi(0) = 1/\sqrt{2\pi}$, $\phi^{(1)}(0) = \phi^{(3)}(0) = 0$, $\phi^{(2)}(0) = -1/\sqrt{2\pi}$, and uniform boundedness of $\left|\phi^{(2r)}(\cdot)\right|$ for

$r = 1, 2$, we have

$$\begin{aligned}
&\Phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=\ 0} \\
&= \Phi\left\{b^{1/2}\left(\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right)(1+O\,(b))\right\} \\
&= \Phi\,(0)+\phi\,(0)\,b^{1/2}\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}\{1+O\,(b)\} \\
&+\frac{1}{2}\phi^{(1)}\,(0)\,b\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}^2\{1+O\,(b)\}^2+O\left(b^{3/2}\right) \\
&= \frac{1}{2}+\frac{(m+2)\,t_T-(m+1)}{\sqrt{2\pi}\sqrt{t_T\,(1-t_T)}}b^{1/2}+O\left(b^{3/2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} \\
&= \phi\left\{b^{1/2}\left(\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right)(1+O\,(b))\right\} \\
&= \phi\,(0)+\phi^{(1)}\,(0)\,b^{1/2}\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}\{1+O\,(b)\} \\
&+\frac{1}{2}\phi^{(2)}\,(0)\,b\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}^2\{1+O\,(b)\}^2+O\left(b^2\right) \\
&= \frac{1}{\sqrt{2\pi}}\left[1-\frac{\{(m+2)\,t_T-(m+1)\}^2}{2t_T\,(1-t_T)}b+O\left(b^2\right)\right]. \qquad \text{(S22)}
\end{aligned}$$

It can be also shown that

$$\begin{aligned}
&\frac{\partial A^{\pm}_{b,t_T,m}}{\partial\Delta}\bigg|_{\Delta=0} \\
&= \mp\frac{b^{-1/2}}{\sqrt{t_T\,(1-t_T)}}\left\{1+\frac{2t_T^2-2t_T+1+m\,(1-t_T)^2}{t_T\,(1-t_T)}b+O\left(b^2\right)\right\}. \qquad \text{(S23)}
\end{aligned}$$

A straightforward but tedious calculation finally delivers

$$\begin{aligned}
\Phi\left(A^{\pm}_{b,t_T,m}\right) &= \frac{1}{2}\mp\frac{1}{\sqrt{2\pi}\sqrt{t_T\,(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right)+\frac{(2t_T-1)-m\,(1-t_T)}{\sqrt{2\pi}\sqrt{t_T\,(1-t_T)}}b^{1/2} \\
&\mp\left[\frac{m}{\sqrt{2\pi}\sqrt{t_T\,(1-t_T)}}+\frac{1-m^2\,(1-t_T)^2}{2\sqrt{2\pi}\,\{t_T\,(1-t_T)\}^{3/2}}+\right]\Delta b^{1/2} \\
&+O\left\{\max\left(b^{3/2},\frac{\Delta^2}{b}\right)\right\},
\end{aligned}$$

which establishes (A2).

Proof of (A3). It follows from the identity $\phi^{(1)}(v) \equiv -v\phi(v)$ that $C^{\pm}_{b,t_T,m}(1) = \phi\left(B^{\pm}_{b,t_T,m}\right) - \phi\left(A^{\pm}_{b,t_T,m}\right)$. Since $\left|B^{\pm}_{b,t_T,m}\right| \leq O\left(b^{-1/2}\right)$ is shown in the proof of Proposition 2(i), it is the case that $\phi\left(B^{\pm}_{b,t_T,m}\right) = \phi\left(\left|B^{\pm}_{b,t_T,m}\right|\right) \to 0$ at an exponential rate. Therefore, again we may focus only on $\phi\left(A^{\pm}_{b,t_T,m}\right)$. By a second-order Taylor expansion of $\phi\left(A^{\pm}_{b,t_T,m}\right)$ around $\Delta = 0$,

$$\phi\left(A^{\pm}_{b,t_T,m}\right) = \phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} + \phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\frac{\partial A^{\pm}_{b,t_T,m}}{\partial \Delta}\bigg|_{\Delta=0}\Delta + O\left(\Delta^2\right).$$

It follows from (S22) that $\phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} = \left(1/\sqrt{2\pi}\right)\{1+O(b)\}$. Then, by $\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right) = -A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right)$ and (S21),

$$\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} = -b^{1/2}\left\{\frac{(m+2)t_T-(m+1)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}}\right\}\{1+O(b)\}. \tag{S24}$$

Therefore, by (S22)-(S24) and $\Delta^2 = o(b)$,

$$\phi\left(A^{\pm}_{b,t_T,m}\right) = \frac{1}{\sqrt{2\pi}} \pm \frac{(m+2)t_T-(m+1)}{\sqrt{2\pi}t_T(1-t_T)}\Delta + O(b), \tag{S25}$$

and thus (A3) immediately follows.

Proof of (A4). By integration by parts,

$$C^{\pm}_{b,t_T,m}(3) = \left(B^{\pm}_{b,t_T,m}\right)^2\phi\left(B^{\pm}_{b,t_T,m}\right) - \left(A^{\pm}_{b,t_T,m}\right)^2\phi\left(A^{\pm}_{b,t_T,m}\right) + 2\left\{\phi\left(B^{\pm}_{b,t_T,m}\right) - \phi\left(A^{\pm}_{b,t_T,m}\right)\right\},$$

where each of $\left(B^{\pm}_{b,t_T,m}\right)^2\phi\left(B^{\pm}_{b,t_T,m}\right)$ and $\phi\left(B^{\pm}_{b,t_T,m}\right)$ converges to zero at an exponential rate, and an approximation to $\phi\left(A^{\pm}_{b,t_T,m}\right)$ has been already derived as (S25). The remaining task is to demonstrate that

$$\left(A^{\pm}_{b,t_T,m}\right)^2\phi\left(A^{\pm}_{b,t_T,m}\right) = \mp\frac{2\{(m+2)t_T-(m+1)\}}{\sqrt{2\pi}t_T(1-t_T)}\Delta + O(b), \tag{S26}$$

because if this is true, then (A4) is immediately shown. By a second-order Taylor expansion of $\left(A^{\pm}_{b,t_T,m}\right)^2\phi\left(A^{\pm}_{b,t_T,m}\right)$ around $\Delta = 0$,

$$\begin{aligned}
&\left(A^{\pm}_{b,t_T,m}\right)^2\phi\left(A^{\pm}_{b,t_T,m}\right) \\
&= \left(A^{\pm}_{b,t_T,m}\right)^2\phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} \\
&+ \left\{2A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right) + \left(A^{\pm}_{b,t_T,m}\right)^2\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\right\}\frac{\partial A^{\pm}_{b,t_T,m}}{\partial\Delta}\bigg|_{\Delta=0}\Delta \\
&+ O\left(\Delta^2\right).
\end{aligned} \tag{S27}$$

By (S21)-(S24), $A^{\pm}_{b,t_T,m}\Big|_{\Delta=\,0} = O\left(b^{1/2}\right)$, $\phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=\,0} = O\left(1\right)$, $\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=\,0} = O\left(b^{1/2}\right)$, and $\partial A^{\pm}_{b,t_T,m}/\partial\Delta\Big|_{\Delta=\,0} = O\left(b^{-1/2}\right)$. It follows from $\Delta^2 = o\left(b\right)$ that (S27) can be simplified further as

$$\left(A^{\pm}_{b,t_T,m}\right)^2 \phi\left(A^{\pm}_{b,t_T,m}\right) = 2A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right)\left.\frac{\partial A^{\pm}_{b,t_T,m}}{\partial\Delta}\right|_{\Delta=0}\Delta + O\left(b\right).$$

By $A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right) = -\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)$, (S23) and (S24), we have

$$\begin{aligned} A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right)\left.\frac{\partial A^{\pm}_{b,t_T,m}}{\partial\Delta}\right|_{\Delta=0} &= -\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\left.\frac{\partial A^{\pm}_{b,t_T,m}}{\partial\Delta}\right|_{\Delta=0} \\ &= \mp\left\{\frac{\left(m+2\right)t_T - \left(m+1\right)}{\sqrt{2\pi}t_T\left(1-t_T\right)}\right\}\left\{1+O\left(b\right)\right\}, \end{aligned}$$

which implies (S26). This completes the proof. ■

S.2.4 Proof of Lemma A5

By definition, $E\left\{\hat{J}^{(1)}\left(t_T\right)\right\} = E\left\{\dot{K}^{-}_{t_T}\left(Y_i\right)\right\} - E\left\{\dot{K}^{+}_{t_T}\left(Y_i\right)\right\}$, where, by (4),

$$\begin{aligned} E\left\{\dot{K}^{\pm}_{t_T}\left(Y_i\right)\right\} &= \int_0^1 \dot{K}^{\pm}_{t_T}\left(u\right) f_Y\left(y\right)dy \\ &= \int_0^1 \dot{K}^{\pm}_{t_T}\left(u\right) g_Y\left(u\right)du + d_T\int_0^{t_T}\dot{K}^{\pm}_{t_T}\left(u\right)du \\ &=: E^{\pm}_{g_Y} + E^{\pm}_{d_T}. \end{aligned}$$

For $\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)$ defined in Lemma A2, $\dot{K}^{\pm}_{t_T}\left(u\right)$ can be expressed as

$$\dot{K}^{\pm}_{t_T}\left(u\right) := \frac{1}{b}\mathcal{L}^{\pm}_{B}\left(u\right)K^{\pm}_{t_T}\left(u\right) := \frac{1}{b}\left\{\ln\left(\frac{u}{1-u}\right) - \Psi\left(p^{\pm}_{b,t_T,0}\right) + \Psi\left(q^{\pm}_{b,t_T}\right)\right\}K^{\pm}_{t_T}\left(u\right).$$

Before proceeding, it is worth remarking how $\mathcal{L}^{\pm}_{B}\left(u\right)$ relates to $L_{B(x,b)}\left(u\right)$ defined in Funke and Hirukawa (2024). Let

$$\mathcal{L}^{0}_{B}\left(u\right) := \mathcal{L}^{\pm}_{B}\left(u\right)\big|_{\Delta=0} = \ln\left(\frac{u}{1-u}\right) - \Psi\left(p^{0}_{b,t_T,0}\right) + \Psi\left(q^{0}_{b,t_T}\right). \tag{S28}$$

Then, it holds that $\mathcal{L}^{0}_{B}\left(u\right) = bL_{B(t_T,b)}\left(u\right)$.

Now $E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}$ can be decomposed into

$$E\left\{\hat{J}^{(1)}\left(t_T\right)\right\} = \left(E^{-}_{g_Y} - E^{+}_{g_Y}\right) + \left(E^{-}_{d_T} - E^{+}_{d_T}\right).$$

A minor modification of the proof for Theorem 2.1(i)(b) of Funke and Hirukawa (2024) gives $E^{\pm}_{g_Y} = g^{(1)}_{Y}\left(t_T\right) \pm g^{(2)}_{Y}\left(t_T\right)\Delta + o\left(\Delta\right)$. As a consequence, $\left|E^{-}_{g_Y} - E^{+}_{g_Y}\right|$ is at most $O\left(\Delta\right) = o\left(\Delta/b^{1/2}\right)$. In what follows, it suffices to demonstrate that

$$E^{-}_{d_T} - E^{+}_{d_T} = \frac{d_T\left(2-t_T\right)}{\sqrt{2\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right) + o\left(\frac{\Delta}{b^{1/2}}\right). \tag{S29}$$

After substituting

$$\ln\left(\frac{u}{1-u}\right)=\ln\left(\frac{t_T}{1-t_T}\right)+\left\{\frac{1}{t_T\left(1-t_T\right)}\right\}\left(u-t_T\right)$$
$$+\left\{\frac{2t_T-1}{2t_T^2\left(1-t_T\right)^2}\right\}\left(u-t_T\right)^2+o\left(\left|u-t_T\right|^2\right),$$
$$\Psi\left(p_{b,t_T,0}^{\pm}\right)=\ln\left(\frac{t_T\pm\Delta}{b}\right)+\frac{b}{2\left(t_T\pm\Delta\right)}+O\left(b^2\right),$$

and

$$\Psi\left(q_{b,t_T}^{\pm}\right)=\ln\left\{\frac{1-\left(t_T\pm\Delta\right)}{b}\right\}+\frac{b}{2\left\{1-\left(t_T\pm\Delta\right)\right\}}+O\left(b^2\right)$$

by (S5) into $\mathcal{L}_B^{\pm}(u)$, using

$$\ln\left(t_T\pm\Delta\right)-\ln t_T=\pm\frac{\Delta}{t_T}-\frac{\Delta^2}{2t_T^2}\pm\frac{\Delta^3}{3t_T^3}+O\left(\Delta^4\right),$$
$$\ln\left\{1-\left(t_T\pm\Delta\right)\right\}-\ln\left(1-t_T\right)=\mp\frac{\Delta}{1-t_T}-\frac{\Delta^2}{2\left(1-t_T\right)^2}\mp\frac{\Delta^3}{3\left(1-t_T\right)^2}+O\left(\Delta^4\right),$$

and

$$\frac{1}{1-\left(t_T\pm\Delta\right)}-\frac{1}{t_T\pm\Delta}=\frac{2t_T-1}{t_T\left(1-t_T\right)}\pm\frac{2t_T^2-2t_T+1}{t_T^2\left(1-t_T\right)^2}\Delta+O\left(\Delta^2\right),$$

and recognizing that $O\left(\Delta^4\right)$ and $O\left(\Delta^2 b\right)$ terms are at most $o\left(b^2\right)$, $\mathcal{L}_B^{\pm}(u)$ can be rearranged as

$$\mathcal{L}_B^{\pm}(u):=L^{\pm}+\left\{\frac{1}{t_T\left(1-t_T\right)}\right\}\left(u-t_T\right)+\left\{\frac{2t_T-1}{2t_T^2\left(1-t_T\right)^2}\right\}\left(u-t_T\right)^2,$$

where

$$L^{\pm}=\mp\frac{\Delta}{t_T\left(1-t_T\right)}-\frac{2t_T-1}{t_T^2\left(1-t_T\right)^2}\Delta^2\mp\frac{3t_T^2-3t_T+1}{3t_T^3\left(1-t_T\right)^3}\Delta^3$$
$$+\frac{2t_T-1}{2t_T\left(1-t_T\right)}b\pm\frac{2t_T^2-2t_T+1}{2t_T^2\left(1-t_T\right)^2}\Delta b+O\left(b^2\right). \tag{S30}$$

It follows that

$$E_{d_T}^{\pm}=\frac{d_T}{b}\left[L^{\pm}\int_0^{t_T}K_{t_T}^{\pm}(u)\,du+\left\{\frac{1}{t_T\left(1-t_T\right)}\right\}\int_0^{t_T}\left(u-t_T\right)K_{t_T}^{\pm}(u)\,du\right.$$
$$\left.+\left\{\frac{2t_T-1}{2t_T^2\left(1-t_T\right)^2}\right\}\int_0^{t_T}\left(u-t_T\right)^2K_{t_T}^{\pm}(u)\,du\right]. \tag{S31}$$

Now

$$\int_0^{t_T}u^mK_{t_T}^{\pm}(u)\,du=\frac{B\left(p_{b,t_T,m}^{\pm},q_{b,t_T}^{\pm}\right)}{B\left(p_{b,t_T,0}^{\pm},q_{b,t_T}^{\pm}\right)}\int_0^{t_T}f_{Z_{b,t_T,m}^{\pm}}(z)\,dz,$$

where $f_{Z^{\pm}_{b,t_T,m}}(z)$ is the probability density function (pdf) of the beta random variable $Z^{\pm}_{b,t_T,m}$, and

$$\frac{B\left(p^{\pm}_{b,t_T,m}, q^{\pm}_{b,t_T}\right)}{B\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)} = \begin{cases} 1 & \text{for } m = 0 \\ \dfrac{\prod_{k=1}^{m}\left(\frac{t_T \pm \Delta}{b} + k\right)}{\prod_{k=1}^{m}\left(\frac{1}{b} + k + 1\right)} & \text{for } m \geq 1 \end{cases}.$$

by the property of the beta function. Applying Lemmata A2 and A4 and then making straightforward but tedious calculations, we also have, as $n \to \infty$,

$$\int_0^{t_T} f_{Z^{\pm}_{b,t_T,m}}(z)\,dz := \Lambda^{\pm}_b - \frac{m(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \mp \frac{m(2-m)(1-t_T)^2}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}} \Delta b^{1/2}, \quad \text{(S32)}$$

where

$$\begin{aligned} \Lambda^{\pm}_b &= \frac{1}{2} \mp \frac{1}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} \left(\frac{\Delta}{b^{1/2}}\right) + \frac{2(2t_T-1)}{3\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \\ &\mp \frac{2+3(2t_T-1)^2}{3\sqrt{2\pi}(t_T(1-t_T))^{3/2}} \Delta b^{1/2} + O\left(\frac{\Delta^2}{b}\right) + O(b) + O\left(b^{3/2}\right). \end{aligned} \quad \text{(S33)}$$

The above results provide approximations to $\int_0^{t_T} u^m K^{\pm}_{t_T}(u)\,du$ for $m = 0, 1, 2$. Obviously, for $m = 0$,

$$\int_0^{t_T} K^{\pm}_{t_T}(u)\,du = \int_0^{t_T} f_{Z^{\pm}_{b,t_T,0}}(z)\,dz = \Lambda^{\pm}_b. \quad \text{(S34)}$$

For $m = 1$,

$$\begin{aligned} &\int_0^{t_T} u K^{\pm}_{t_T}(u)\,du \\ &= \frac{(t_T \pm \Delta)/b + 1}{1/b + 2} \int_0^{t_T} f_{Z^{\pm}_{b,t_T,1}}(z)\,dz \\ &= \left\{t_T \pm \Delta - (2t_T - 1)b \mp 2\Delta b + O\left(b^2\right)\right\} \\ &\times \left[\Lambda^{\pm}_b - \frac{1-t_T}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \mp \frac{(1-t_T)^2}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}} \Delta b^{1/2}\right]. \end{aligned}$$

Finally, for $m = 2$,

$$\begin{aligned} &\int_0^{t_T} u^2 K^{\pm}_{t_T}(u)\,du \\ &= \frac{\{(t_T \pm \Delta)/b + 1\}\{(t_T \pm \Delta)/b + 2\}}{(1/b + 2)(1/b + 3)} \int_0^{t_T} f_{Z^{\pm}_{b,t_T,2}}(z)\,dz \\ &= \left\{t_T^2 \pm 2t_T\Delta + \left(3t_T - 5t_T^2\right)b + \Delta^2 \pm (3 - 10t_T)\Delta b + O\left(b^2\right)\right\} \\ &\times \left\{\Lambda^{\pm}_b - \frac{2(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2}\right\}. \end{aligned}$$

After straightforward but tedious calculations, we obtain

$$\begin{aligned}
&\int_0^{t_T} (u - t_T)\, K_{t_T}^{\pm}(u)\, du \\
&= \left\{\pm\Delta - (2t_T - 1)\, b \mp 2\Delta b + O\left(b^2\right)\right\} \Lambda_b^{\pm} \\
&- \frac{t_T(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \mp \frac{2t_T(1-t_T)^2}{3\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}} \Delta b^{1/2} + O\left(b^{3/2}\right)
\end{aligned} \tag{S35}$$

and

$$\begin{aligned}
&\int_0^{t_T} (u - t_T)^2\, K_{t_T}^{\pm}(u)\, du \\
&= \left\{t_T(1-t_T)\, b + \Delta^2 \pm 3(1-2t_T)\,\Delta b + O\left(b^2\right)\right\} \Lambda^{\pm} \\
&\mp \frac{t_T(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} \Delta b^{1/2} + O\left(b^{3/2}\right).
\end{aligned} \tag{S36}$$

Substituting (S30), (S34), (S35), and (S36) into (S31) and simplifying this yield

$$\begin{aligned}
E_{d_T}^{\pm} = \frac{d_T}{b} \Bigg\{ & \underbrace{O\left(\Delta^2\right)}_{=o\left(\Delta b^{1/2}\right)} \mp \underbrace{\frac{1-2t_T}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{5/2}} \left(\frac{\Delta^3}{b^{1/2}}\right)}_{=o\left(\Delta b^{1/2}\right)} + \underbrace{O(\Delta b)}_{=o\left(\Delta b^{1/2}\right)} + \underbrace{O\left(\Delta^3\right)}_{=o\left(\Delta b^{1/2}\right)} \\
& + \underbrace{O\left(b^2\right)}_{=o\left(\Delta b^{1/2}\right)} + O\left(b^{1/2}\right) \mp \frac{2-t_T}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}} \Delta b^{1/2} + \underbrace{O\left(b^{3/2}\right)}_{=o\left(\Delta b^{1/2}\right)} \Bigg\} \\
= \frac{d_T}{b} & \left\{ \mp \frac{2-t_T}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}} \Delta b^{1/2} + o\left(\Delta b^{1/2}\right) + O\left(b^{1/2}\right) \right\}.
\end{aligned}$$

Finally, (S29) can be established by recognizing that $O\left(b^{1/2}\right)$ terms inside the brackets of $E_{d_T}^{\pm}$ are cancelled out after taking the difference $E_{d_T}^{-} - E_{d_T}^{+}$. This completes the proof. ∎

S.2.5 Proof of Lemma A6

Because

$$Var\left\{\hat{J}^{(1)}(t_T)\right\} = \frac{1}{n} Var(H_i) = \frac{1}{n}\left\{E\left(H_i^2\right) - E^2(H_i)\right\},$$

and $|E(H_i)| = \left|E\left\{\hat{J}^{(1)}(t_T)\right\}\right| = O\left(\Delta/b^{1/2}\right)$ as shown above, we concentrate on approximating

$$E\left(H_i^2\right) = \int_0^1 \left\{\dot{K}_{t_T}^{-}(u) - \dot{K}_{t_T}^{+}(u)\right\}^2 f_Y(u)\, du.$$

However, it is not obvious whether the right-hand side may be safely rewritten as

$$\left\{\frac{f_Y(t_T^-) + f_Y(t_T^+)}{2}\right\} \int_0^1 \left\{\dot{K}_{t_T}^{-}(u) - \dot{K}_{t_T}^{+}(u)\right\}^2 du,$$

like the cases in which standard symmetric kernels are employed. It will shortly turn out that this is also valid in our case. Using (4), we decompose $E\left(H_i^2\right)$ into two parts so that

$$\begin{aligned} E\left(H_i^2\right) &= \int_0^1 \left\{\dot{K}_{t_T}^{-}(u) - \dot{K}_{t_0}^{+}(u)\right\}^2 g_Y(u)\, du + d_T \int_0^{t_T} \left\{\dot{K}_{t_T}^{-}(u) - \dot{K}_{t_T}^{+}(u)\right\}^2 du \\ &=: V_{g_Y} + V_{d_T}. \end{aligned}$$

Then, the proof boils down to demonstrating the following two statements:

$$V_{g_Y} \sim g_Y(t_T) \frac{3}{2\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{5/2}} \left(\frac{\Delta^2}{b^{5/2}}\right). \tag{S37}$$

$$V_{d_T} \sim \left(\frac{d_T}{2}\right) \frac{3}{2\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{5/2}} \left(\frac{\Delta^2}{b^{5/2}}\right). \tag{S38}$$

It should be recognized that (S37) and (S38) jointly establish the lemma, because $g_Y(t_T) + d_T/2 = \left\{f_Y(t_T^-) + f_Y(t_T^+)\right\}/2$.

S.2.5.1 Proof of (S37)

V_{g_Y} can be further decomposed into $V_{g_Y} := V_{g_Y}^{2-} + V_{g_Y}^{2+} - 2V_{g_Y}^{+-}$, where

$$V_{g_Y}^{2\pm} = \int_0^1 \left\{\dot{K}_{t_T}^{\pm}(u)\right\}^2 g_Y(u)\, du$$

and

$$V_{g_Y}^{+-} = \int_0^1 \dot{K}_{t_T}^{-}(u)\, \dot{K}_{t_T}^{+}(u)\, g_Y(u)\, du.$$

In what follows, approximations to $V_{g_Y}^{2\pm}$ and $V_{g_Y}^{+-}$ are derived separately, and (S37) is finally obtained.

(i) Approximation to $V_{g_Y}^{2\pm}$. Observe that

$$\begin{aligned} V_{g_Y}^{2\pm} &= \frac{1}{b^2} \int_0^1 \left\{\mathcal{L}_B^{\pm}(u)\, K_{t_T}^{\pm}(u)\right\}^2 g_Y(u)\, du \\ &= \frac{1}{b^2} \int_0^1 \left\{\ln\left(\frac{u}{1-u}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 \left\{K_{t_T}^{\pm}(u)\right\}^2 g_Y(u)\, du, \end{aligned} \tag{S39}$$

where the integral part can be alternatively expressed as

$$G_b^{\pm}(t_T)\, E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 g_Y\left(Z_{b/2,t_T,0}^{\pm}\right)\right]$$

for the beta random variable $Z_{b/2,t_T,m}^{\pm}$ and

$$G_b^{\pm}(t_T) := \frac{B\left(p_{b/2,t_T,0}^{\pm}, q_{b/2,t_T}^{\pm}\right)}{B^2\left(p_{b,t_T,0}^{\pm}, q_{b,t_T}^{\pm}\right)}.$$

Then, by the property of the beta function, definitions of $\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)$, (S1) and $b = o(\Delta)$,

$$G^{\pm}_b(t_T) = \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{(t_T \pm \Delta)\{1-(t_T \pm \Delta)\}}}\{1+O(b)\}$$
$$= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T(1-t_T)}}\{1+O(\Delta)\}. \tag{S40}$$

Furthermore, a mean-value expansion of $g_Y\left(Z^{\pm}_{b/2,t_T,0}\right)$ around $Z^{\pm}_{b/2,t_T,0} = t_T$ leads to

$$E\left[\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right) - \Psi\left(p^{\pm}_{b,t_T,0}\right) + \Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2 g_Y\left(Z^{\pm}_{b/2,t_T,0}\right)\right]$$
$$= g_Y(t_T)\, E\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right) - \Psi\left(p^{\pm}_{b,t_T,0}\right) + \Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2$$
$$+ E\left[\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right) - \Psi\left(p^{\pm}_{b,t_T,0}\right) + \Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2 g^{(1)}_Y\left(\bar{t}^{\pm}_T\right)\left(Z^{\pm}_{b/2,t_T,0} - t_T\right)\right]$$
$$=: W^{\pm}_1 + W^{\pm}_2$$

for some $\bar{t}^{\pm}_T$ on the line segment joining $Z^{\pm}_{b/2,t_T,0}$ and t_T. We work on $W^{\pm}_1$ first. It follows from definitions of $\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)$ and (S3) that

$$W^{\pm}_1 = g_Y(t_T)\left[\left\{\Psi\left(\frac{2(t_T \pm \Delta)}{b}+1\right) - \Psi\left(\frac{2(1-(t_T \pm \Delta))}{b}+1\right)\right.\right.$$
$$\left. -\Psi\left(\frac{t_T \pm \Delta}{b}+1\right) + \Psi\left(\frac{1-(t_T \pm \Delta)}{b}+1\right)\right\}^2$$
$$\left. +\Psi^{(1)}\left\{\frac{2(t_T \pm \Delta)}{b}+1\right\} + \Psi^{(1)}\left\{\frac{2(1-(t_T \pm \Delta))}{b}+1\right\}\right].$$

Then, by (S5) and (S6),

$$\left[\Psi\left\{\frac{2(t_T \pm \Delta)}{b}+1\right\} - \Psi\left\{\frac{2(1-(t_T \pm \Delta))}{b}+1\right\}\right.$$
$$\left. -\Psi\left(\frac{t_T \pm \Delta}{b}+1\right) + \Psi\left\{\frac{1-(t_T \pm \Delta)}{b}+1\right\}\right]^2 = O\left(b^2\right),$$

and

$$\Psi^{(1)}\left\{\frac{2(t_T \pm \Delta)}{b}+1\right\} + \Psi^{(1)}\left\{\frac{2(1-(t_T \pm \Delta))}{b}+1\right\}$$
$$= \frac{b}{2(t_T \pm \Delta)\{1-(t_T \pm \Delta)\}}\{1+O(b)\}.$$

It follows from $b = o(\Delta)$ that

$$W^{\pm}_1 = \frac{g_Y(t_T)}{2(t_T \pm \Delta)\{1-(t_T \pm \Delta)\}} b\{1+O(b)\} = \frac{b g_Y(t_T)}{2t_T(1-t_T)}\{1+O(\Delta)\}.$$

The remaining task is to show that $\left|W_2^{\pm}\right|$ is of smaller order in magnitude than $\left|W_1^{\pm}\right|$. To be more precise, we demonstrate that $\left|W_2^{\pm}\right| = O\left(b^{3/2}\right)$. To see this, observe that

$$\begin{aligned}
&\left|W_2^{\pm}\right| \\
&\leq E\left\{\left|\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right|^2 \left|g_Y^{(1)}\left(\bar{t}_T^{\pm}\right)\right| \left|Z_{b/2,t_T,0}^{\pm} - t_T\right|\right\} \\
&\leq \sup_{y\in[0,1]}\left|g_Y^{(1)}(y)\right| \left[E\left\{\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2\right]^{1/2} \\
&\times \left[E\left\{\left(\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right)^2 \left(Z_{b/2,t_T,0}^{\pm} - t_T\right)^2\right\}\right]^{1/2}
\end{aligned}$$

by the Cauchy-Schwarz inequality. Now $\sup_{y\in[0,1]}\left|g_Y^{(1)}(y)\right| < \infty$ by Assumption 2, and $E\left\{\ln\left(Z_{b/2,t_T,0}^{\pm}/\left(1-Z_{b/2,t_T,0}^{\pm}\right)\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 = O(b)$ as demonstrated above. By (S2)-(S6), we can also obtain

$$E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 \left(Z_{b/2,t_T,0}^{\pm} - t_T\right)^2\right] = O\left(b^2\right),$$

and thus $\left|W_2^{\pm}\right| = O\left(b^{3/2}\right)$ follows.

Therefore, by $\Delta = o\left(b^{1/2}\right)$,

$$\begin{aligned}
&E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 g_Y\left(Z_{b/2,t_T,0}^{\pm}\right)\right] \\
&= \frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 + O(\Delta)\right\} + O\left(b^{3/2}\right) \\
&= \frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 + O\left(b^{1/2}\right)\right\}. \qquad \text{(S41)}
\end{aligned}$$

Substituting (S40) and (S41) into (S39) and recognizing that $\Delta = o\left(b^{1/2}\right)$, we may conclude that

$$\begin{aligned}
V_{g_Y}^{2\pm} &= \frac{1}{b^2}\left[\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left\{1 + O(\Delta)\right\}\right]\left[\frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 + O\left(b^{1/2}\right)\right\}\right] \\
&= \frac{b^{-3/2}}{4\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{3/2}}\left\{g_Y(t_T) + O\left(b^{1/2}\right)\right\}.
\end{aligned}$$

(ii) Approximation to $V_{g_Y}^{+-}$. Next, the integral part of

$$V_{g_Y}^{+-} = \frac{1}{b^2}\int_0^1 \mathcal{L}_B^{-}(u)\,\mathcal{L}_B^{+}(u)\,K_{t_T}^{-}(u)\,K_{t_T}^{+}(u)\,g_Y(u)\,du \qquad \text{(S42)}$$

can be rewritten as

$$G_b^0(t_T)\,E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^-_{b,t_T,0}\right)+\Psi\left(q^-_{b,t_T}\right)\right\}\right.$$
$$\left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^+_{b,t_T,0}\right)+\Psi\left(q^+_{b,t_T}\right)\right\}g_Y\left(Z^0_{b/2,t_T,0}\right)\right],$$

for the beta random variable $Z^0_{b/2,t_T,0}$ and

$$G_b^0(t_T):=\frac{B\left(p^0_{b/2,t_T,0},q^0_{b/2,t_T}\right)}{B\left(p^-_{b,t_T,0},q^-_{b,t_T}\right)B\left(p^+_{b,t_T,0},q^+_{b,t_T}\right)}.$$

Then, by the property of the beta function, definitions of $\left(p^\pm_{b,t_T,0},q^\pm_{b,t_T}\right)$ and (S1),

$$\begin{aligned}
&G_b^0(t_T)\\
&=\frac{b^{-1/2}}{2\sqrt{\pi}}\left(\frac{t_T}{t_T^2-\Delta^2}\right)^{1/2}\left\{\frac{1-t_T}{(1-t_T)^2-\Delta^2}\right\}^{1/2}\left(\frac{t_T}{t_T-\Delta}\right)^{(t_T-\Delta)/b}\left(\frac{t_T}{t_T+\Delta}\right)^{(t_T+\Delta)/b}\\
&\times\left(\frac{1-t_T}{1-t_T+\Delta}\right)^{(1-t_T+\Delta)/b}\left(\frac{1-t_T}{1-t_T-\Delta}\right)^{(1-t_T-\Delta)/b}\{1+O(b)\}\\
&=\frac{b^{-1/2}}{2\sqrt{\pi}}\left(\frac{t_T}{t_T^2-\Delta^2}\right)^{1/2}\left\{\frac{1-t_T}{(1-t_T)^2-\Delta^2}\right\}^{1/2}\frac{1}{\left(1-\Delta^2/t_T^2\right)^{t_T/b}}\left(1-\frac{2\Delta}{t_T+\Delta}\right)^{\Delta/b}\\
&\times\frac{1}{\left\{1-\Delta^2/(1-t_T)^2\right\}^{(1-t_T)/b}}\left(1-\frac{2\Delta}{1-t_T+\Delta}\right)^{\Delta/b}\{1+O(b)\}.
\end{aligned}$$

Because

$$\begin{aligned}
\left(\frac{t_T}{t_T^2-\Delta^2}\right)^{1/2}\left\{\frac{1-t_T}{(1-t_T)^2-\Delta^2}\right\}^{1/2}&=\frac{1+O\left(\Delta^2\right)}{\sqrt{t_T(1-t_T)}},\\
\frac{1}{\left(1-\Delta^2/t_T^2\right)^{t_T/b}}\left(1-\frac{2\Delta}{t_T+\Delta}\right)^{\Delta/b}&=\frac{1-(2/t_T)\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)}{1-(1/t_T)\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)}\\
&=1-\frac{1}{t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\left\{1-\Delta^2/(1-t_T)^2\right\}^{(1-t_T)/b}}\left(1-\frac{2\Delta}{1-t_T+\Delta}\right)^{\Delta/b}&=\frac{1-\{2/(1-t_T)\}\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)}{1-\{1/(1-t_T)\}\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)}\\
&=1-\frac{1}{1-t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right),
\end{aligned}$$

and $b = o\left(\Delta^2/b\right)$, we have

$$
\begin{aligned}
G_b^0\left(t_T\right) &= \frac{b^{-1/2}\left\{1+O\left(\Delta^2\right)\right\}}{2\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left\{1-\frac{1}{t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\
&\times\left\{1-\frac{1}{1-t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}\left\{1+O\left(b\right)\right\} \\
&= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left\{1-\frac{1}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}. \qquad \text{(S43)}
\end{aligned}
$$

Moreover, by a mean-value expansion of $g_Y\left(Z_{b/2,t_T,0}^0\right)$ around $Z_{b/2,t_T,0}^0 = t_T$,

$$
\begin{aligned}
& E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^-\right)+\Psi\left(q_{b,t_T}^-\right)\right\}\right. \\
& \left.\times\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^+\right)+\Psi\left(q_{b,t_T}^+\right)\right\}g_Y\left(Z_{b/2,t_T,0}^0\right)\right] \\
& = g_Y\left(t_T\right)E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^-\right)+\Psi\left(q_{b,t_T}^-\right)\right\}\right. \\
& \left.\times\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^+\right)+\Psi\left(q_{b,t_T}^+\right)\right\}\right] \\
& + E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^-\right)+\Psi\left(q_{b,t_T}^-\right)\right\}\right. \\
& \left.\times\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^+\right)+\Psi\left(q_{b,t_T}^+\right)\right\}g_Y^{(1)}\left(\bar{t}_T^0\right)\left(Z_{b/2,t_T,0}^0-t_T\right)\right] \\
& =: W_1^0+W_2^0
\end{aligned}
$$

for some $\bar{t}_T^0$ on the line segment joining $Z_{b/2,t_T,0}^0$ and t_T. For W_1^0, by (S2), (S3), (S5), and (S6),

$$
\begin{aligned}
& E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^-\right)+\Psi\left(q_{b,t_T}^-\right)\right\}\left\{\ln\left(\frac{Z_{b/2,t_T,0}^0}{1-Z_{b/2,t_T,0}^0}\right)-\Psi\left(p_{b,t_T,0}^+\right)+\Psi\left(q_{b,t_T}^+\right)\right\}\right] \\
& = \left[\Psi\left(\frac{2t_T}{b}+1\right)-\Psi\left\{\frac{2\left(1-t_T\right)}{b}+1\right\}-\Psi\left(\frac{t_T-\Delta}{b}+1\right)+\Psi\left\{\frac{1-\left(t_T-\Delta\right)}{b}+1\right\}\right] \\
& \times\left[\Psi\left(\frac{2t_T}{b}+1\right)-\Psi\left\{\frac{2\left(1-t_T\right)}{b}+1\right\}-\Psi\left(\frac{t_T+\Delta}{b}+1\right)+\Psi\left\{\frac{1-\left(t_T+\Delta\right)}{b}+1\right\}\right] \\
& + \Psi^{(1)}\left(\frac{2t_T}{b}+1\right)+\Psi^{(1)}\left\{\frac{2\left(1-t_T\right)}{b}+1\right\} \\
& = \frac{b}{2t_T\left(1-t_T\right)}\left\{1-\frac{2}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}.
\end{aligned}
$$

Therefore,

$$
W_1^0 = \frac{bg_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1-\frac{2}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}.
$$

It can be also shown that $\left|W_2^0\right| = O\left(b^{3/2}\right)$, and thus

$$
\begin{aligned}
&E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right) - \Psi\left(p^-_{b,t_T,0}\right) + \Psi\left(q^-_{b,t_T}\right)\right\}\right. \\
&\quad \left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right) - \Psi\left(p^+_{b,t_T,0}\right) + \Psi\left(q^+_{b,t_T}\right)\right\} g_Y\left(Z^0_{b/2,t_T,0}\right)\right] \\
&= \frac{b g_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1 - \frac{2}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} + O\left(b^{3/2}\right) \\
&= \frac{b g_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1 - \frac{2}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \qquad \text{(S44)}
\end{aligned}
$$

by recognizing that $b^{1/2} = o\left(\Delta^2/b\right)$. Substituting (S43) and (S44) into (S42) finally yields

$$
\begin{aligned}
V^{+-}_{g_Y} &= \frac{1}{b^2}\left\{\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\right\}\left\{1 - \frac{1}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \\
&\quad \times \frac{b g_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1 - \frac{2}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \\
&= \frac{b^{-3/2}}{4\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left\{g_Y\left(t_T\right) - \frac{3g_Y\left(t_T\right)}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}.
\end{aligned}
$$

(iii) Proof of (S37). Combining two approximations above and again using $b^{1/2} = o\left(\Delta^2/b\right)$ yield

$$
\begin{aligned}
V_{g_Y} &= \left(V^{2-}_{g_Y} + V^{2+}_{g_Y}\right) - 2V^{+-}_{g_Y} \\
&= \frac{b^{-3/2}}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left\{g_Y\left(t_T\right) + O\left(b^{1/2}\right)\right\} \\
&\quad - \frac{b^{-3/2}}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left\{g_Y\left(t_T\right) - \frac{3g_Y\left(t_T\right)}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \\
&= \frac{3g_Y\left(t_T\right)}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right) + o\left(\frac{\Delta^2}{b^{5/2}}\right).
\end{aligned}
$$

Therefore, (S37) is established.

S.2.5.2 Proof of (S38)

Let

$$V^{2\pm}_{d_T} := d_T \int_0^{t_T} \left\{\dot{K}^{\pm}_{t_T}\left(u\right)\right\}^2 du$$

and

$$V^{+-}_{d_T} := d_T \int_0^{t_T} \dot{K}^{-}_{t_T}\left(u\right)\dot{K}^{+}_{t_T}\left(u\right) du.$$

Then, it holds that $V_{d_T} = V^{2-}_{d_T} + V^{2+}_{d_T} - 2V^{+-}_{d_T}$. As above, $V^{2\pm}_{d_T}$ and $V^{+-}_{d_T}$ are approximated separately.

(i) Approximation to $V_{d_T}^{2\pm}$. Given the notation in the proof of (S37), $V_{d_T}^{2\pm}$ can be expressed as

$$
\begin{aligned}
V_{d_T}^{2\pm} &= \frac{d_T}{b^2}\int_0^{t_T}\left\{\mathcal{L}_B^{\pm}(u)\,K_{t_T}^{\pm}(u)\right\}^2 du \\
&= \frac{d_T G_b^{\pm}(t_T)}{b^2}\int_0^{t_T}\left\{\mathcal{L}_B^{\pm}(z)\right\}^2 f_{Z_{b/2,t_T,0}^{\pm}}(z)\,dz, \qquad \text{(S45)}
\end{aligned}
$$

where $f_{Z_{b/2,t_T,m}^{\pm}}(z)$ is the pdf of the beta random variable $Z_{b/2,t_T,m}^{\pm}$. Below the integral part is approximated. After substituting

$$
\begin{aligned}
&\ln\left(\frac{z}{1-z}\right) \\
&= \ln\left(\frac{t_T}{1-t_T}\right) + \left\{\frac{1}{t_T(1-t_T)}\right\}(z-t_T) + \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}(z-t_T)^2 \\
&+ o\left(|z-t_T|^2\right), \\
&\ln^2\left(\frac{z}{1-z}\right) \\
&= \ln^2\left(\frac{t_T}{1-t_T}\right) + \left[\frac{2\ln\{t_T/(1-t_T)\}}{t_T(1-t_T)}\right](z-t_T) \\
&+ \left[\frac{1+(2t_T-1)\ln\{t_T/(1-t_T)\}}{t_T^2(1-t_T)^2}\right](z-t_T)^2 + o\left(|z-t_T|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Psi\left(p_{b,t_T,0}^{+}\right) - \Psi\left(q_{b,t_T}^{+}\right) \\
&= \ln\left(\frac{t_T}{1-t_T}\right) \mp \frac{\Delta}{t_T(1-t_T)} + \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}\Delta^2 - \left\{\frac{2t_T-1}{2t_T(1-t_T)}\right\}b + O(\Delta b)
\end{aligned}
$$

by (S5) into $\left\{\mathcal{L}_B^{\pm}(z)\right\}^2$ and making straightforward but tedious calculations, we can rewrite $\left\{\mathcal{L}_B^{\pm}(z)\right\}^2$ as

$$
\begin{aligned}
&\left\{\mathcal{L}_B^{\pm}(z)\right\}^2 \\
&= \left[\mp\frac{2\Delta}{t_T^2(1-t_T)^2} + \left\{\frac{2t_T-1}{t_T^2(1-t_T)^2}\right\}b + O\left(\Delta^2\right)\right](z-t_T) \\
&+ \left[\frac{1}{t_T^2(1-t_T)^2} \mp \left\{\frac{2t_T-1}{t_T^3(1-t_T)^3}\right\}\Delta + \left\{\frac{(2t_T-1)^2}{2t_T^3(1-t_T)^3}\right\}b + O\left(\Delta^2\right)\right](z-t_T)^2 \\
&+ \frac{\Delta^2}{t_T^2(1-t_T)^2} + o\left(\Delta^2\right). \qquad \text{(S46)}
\end{aligned}
$$

It follows from (S32) and (S33) that for $m \geq 0$,

$$
\int_0^{t_T} f_{Z_{b/2,t_T,m}^{\pm}}(z)\,dz = \frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right). \qquad \text{(S47)}
$$

Then, as in the proof of Lemma A5, approximations to $\int_0^{t_T} z^m f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz$ for $m=1,2$ can be derived. For $m=1$,

$$\begin{aligned}&\int_0^{t_T} z f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz\\&=\left\{t_T \pm \Delta + \left(\frac{1-2t_T}{2}\right)b \mp \Delta b + O\left(b^2\right)\right\}\\&\times\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}.\end{aligned}$$

For $m=2$,

$$\begin{aligned}&\int_0^{t_T} z^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz\\&=\left\{t_T^2 \pm 2t_T\Delta + \left(\frac{3t_T-5t_T^2}{2}\right)b + \Delta^2 \pm \left(\frac{3-10t_T}{2}\right)\Delta b + O\left(b^2\right)\right\}\\&\times\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^{t_T} (z-t_T) f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz\\&=\left\{\pm\Delta + \left(\frac{1-2t_T}{2}\right)b \mp \Delta b + O\left(b^2\right)\right\}\\&\times\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\},\end{aligned} \tag{S48}$$

and

$$\begin{aligned}&\int_0^{t_T} (z-t_T)^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz\\&=\left[\left\{\frac{t_T(1-t_T)}{2}\right\}b + \Delta^2 \pm \left\{\frac{3(1-2t_T)}{2}\right\}\Delta b + O\left(b^2\right)\right]\\&\times\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}.\end{aligned} \tag{S49}$$

Combining (S46)-(S49) and using the fact that $O\left(b^{1/2}\right)$ and $O(\Delta)$ terms are at most $o\left(\Delta^2/b\right)$ offer

$$\begin{aligned}&\int_0^{t_T} \left\{\mathcal{L}_B^{\pm}(z)\right\}^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz\\&=\frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}.\end{aligned} \tag{S50}$$

In the end,

$$
\begin{aligned}
V_{d_T}^{2\pm} &= \frac{d_T}{b^2}\left\{\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{(t_T\pm\Delta)\{1-(t_T\pm\Delta)\}}}\right\}\{1+O(b)\} \\
&\times \frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2}\mp\frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\
&= \frac{b^{-3/2}d_T}{4\sqrt{\pi}\{t_T(1-t_T)\}^{3/2}}\left\{\frac{1}{2}\mp\frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \qquad \text{(S51)}
\end{aligned}
$$

can be obtained by substituting (S40) and (S50) into (S45) and recognizing that $O(b)$ and $O(\Delta)$ terms are at most $o\left(\Delta^2/b\right)$.

(ii) Approximation to $V_{d_T}^{+-}$. Our next task is to find an approximation to

$$
\begin{aligned}
V_{d_T}^{+-} &= \frac{d_T}{b^2}\int_0^{t_T}\mathcal{L}_B^{-}(u)\,\mathcal{L}_B^{+}(u)\,K_{t_T}^{-}(u)\,K_{t_T}^{+}(u)\,du \\
&= \frac{d_T G_b^0(t_T)}{b^2}\int_0^{t_T}\mathcal{L}_B^{-}(z)\,\mathcal{L}_B^{+}(z)\,f_{Z^0_{b/2,t_T,0}}(z)\,dz, \qquad \text{(S52)}
\end{aligned}
$$

where $f_{Z^0_{b/2,t_T,0}}(z)$ is the pdf of the beta random variable $Z^0_{b/2,t_T,0}$. By a similar argument as above, it can be shown that

$$
\begin{aligned}
&\mathcal{L}_B^{-}(z)\,\mathcal{L}_B^{+}(z) \\
&= \left[\left\{\frac{2t_T-1}{2t_T(1-t_T)}\right\}b-\left\{\frac{2t_T-1}{t_T^3(1-t_T)^3}\right\}\Delta^2\right](z-t_T) \\
&+ \left[\frac{1}{t_T^2(1-t_T)^2}+\left\{\frac{(2t_T-1)^2}{2t_T^3(1-t_T)^3}\right\}b-\left\{\frac{(2t_T-1)^2}{2t_T^4(1-t_T)^4}\right\}\Delta^2\right](z-t_T)^2 \\
&- \frac{\Delta^2}{t_T^2(1-t_T)^2}+O(\Delta b).
\end{aligned}
$$

Because

$$
\int_0^{t_T} f_{Z^0_{b/2,t_T,0}}(z)\,dz = \frac{1}{2}+O\left(b^{1/2}\right),
$$

$$
\int_0^{t_T}(z-t_T)\,f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz = \left(\frac{1-2t_T}{2}\right)b\left\{\frac{1}{2}+O\left(b^{1/2}\right)\right\},
$$

and

$$
\int_0^{t_T}(z-t_T)^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz = \left\{\frac{t_T(1-t_T)}{2}\right\}b\left\{\frac{1}{2}+O\left(b^{1/2}\right)\right\},
$$

the integral part of (S52) can be simplified as

$$
\begin{aligned}
&\int_0^{t_T}\mathcal{L}_B^{-}(z)\,\mathcal{L}_B^{+}(z)\,f_{Z^0_{b/2,t_T,0}}(z)\,dz \\
&= \frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2}-\frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \qquad \text{(S53)}
\end{aligned}
$$

Substituting (S43) and (S53) into (S52), we can conclude that

$$\begin{aligned}V_{d_T}^{+-} &= \frac{d_T}{b^2}\left\{\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T(1-t_T)}}\right\}\left\{1-\frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\ &\times \frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2}-\frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\ &= \frac{b^{-3/2}d_T}{4\sqrt{\pi}\{t_T(1-t_T)\}^{3/2}}\left\{\frac{1}{2}-\frac{3}{2t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}. \end{aligned} \tag{S54}$$

(iii) Proof of (S38). Combining (S51) and (S54) leads to

$$\begin{aligned}V_{d_T} &= \left(V_{d_T}^{2-}+V_{d_T}^{2+}\right)-2V_{d_T}^{+-} \\ &= \frac{b^{-3/2}d_T}{4\sqrt{\pi}\{t_T(1-t_T)\}^{3/2}}\left\{1+o\left(\frac{\Delta^2}{b}\right)\right\} \\ &- \frac{b^{-3/2}d_T}{4\sqrt{\pi}\{t_T(1-t_T)\}^{3/2}}\left\{1-\frac{3}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\ &= \left(\frac{d_T}{2}\right)\frac{3}{2\sqrt{\pi}\{t_T(1-t_T)\}^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right)+o\left(\frac{\Delta^2}{b}\right),\end{aligned}$$

which establishes (S38). This completes the proof. ■

S.2.6 Proof of Lemma A7

It holds that

$$\begin{aligned}&\left|\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}(\varsigma)-\left[-\sqrt{\frac{2}{\pi}}\frac{d_T}{\{t_T(1-t_T)\}^{3/2}}\right]\right| \\ &\le \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(\varsigma)-\hat{J}^{(2)}(t_T)\right|+\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(t_T)-E\left\{\hat{J}^{(2)}(t_T)\right\}\right| \\ &+\left|\left(\frac{b^{3/2}}{\Delta}\right)E\left\{\hat{J}^{(2)}(t_T)\right\}-\left[-\sqrt{\frac{2}{\pi}}\frac{d_T}{\{t_T(1-t_T)\}^{3/2}}\right]\right| \\ &=: D_1+D_2+D_3.\end{aligned}$$

Then, the proof boils down to establishing the following three statements:

$$D_1 = \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(\varsigma)-\hat{J}^{(2)}(t_T)\right| = o_p(1). \tag{S55}$$

$$D_2 = \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(t_T)-E\left\{\hat{J}^{(2)}(t_T)\right\}\right| = o_p(1). \tag{S56}$$

$$D_3 = \left|\left(\frac{b^{3/2}}{\Delta}\right)E\left\{\hat{J}^{(2)}(t_T)\right\}-\left[-\sqrt{\frac{2}{\pi}}\frac{d_T}{\{t_T(1-t_T)\}^{3/2}}\right]\right| = o(1). \tag{S57}$$

We work on (S57) first and then proceed to (S55) and (S56).

S.2.6.1 Proof of (S57)

The proof is similar to that of Lemma A5, and thus its outline is presented. Because

$$E\left\{\ddot{K}_{t_T}^{\pm}(Y_i)\right\} = \int_0^1 \ddot{K}_{t_T}^{\pm}(u)\, g_Y(u)\, du + d_T \int_0^{t_T} \ddot{K}_{t_T}^{\pm}(u)\, du =: I_{g_Y}^{\pm} + I_{d_T}^{\pm},$$

where

$$\ddot{K}_{t_T}^{\pm}(u) = \frac{1}{b^2}\left[\left\{\mathcal{L}_B^{\pm}(u)\right\}^2 - \left\{\Psi^{(1)}\left(p_{b,t_T,0}^{\pm}\right) + \Psi^{(1)}\left(q_{b,t_T}^{\pm}\right)\right\}\right] K_{t_T}^{\pm}(u),$$

it holds that

$$E\left\{\hat{J}^{(2)}(t_T)\right\} = E\left\{\ddot{K}_{t_T}^{-}(Y_i)\right\} - E\left\{\ddot{K}_{t_T}^{+}(Y_i)\right\} = \left(I_{g_Y}^{-} - I_{g_Y}^{+}\right) + \left(I_{d_T}^{-} - I_{d_T}^{+}\right).$$

By a similar procedure to the proof of Lemma A5, $I_{g_Y}^{\pm} = g_Y^{(2)}(t_T) \pm g_Y^{(3)}(t_T)\Delta + o(\Delta)$. Then, $\left|I_{g_Y}^{-} - I_{g_Y}^{+}\right|$ is at most $O(\Delta)$ because $\left|g_Y^{(3)}(t_T)\right| < \infty$ by Assumption 2. Following the steps in the proof of Lemma A5 also yields

$$\begin{aligned}
&\left\{\mathcal{L}_B^{\pm}(u)\right\}^2 - \left\{\Psi^{(1)}\left(p_{b,t_T,0}^{\pm}\right) + \Psi^{(1)}\left(q_{b,t_T}^{\pm}\right)\right\} \\
&= \left\{\mp 2\Delta + (2t_T - 1)\, b + O\left(\Delta^2\right)\right\}\left\{\frac{u - t_T}{t_T^2 (1 - t_T)^2}\right\} \\
&+ \left[1 \mp \left\{\frac{2t_T - 1}{t_T(1 - t_T)}\right\}\Delta + \left\{\frac{(2t_T - 1)^2}{2t_T(1 - t_T)}\right\} b + O\left(\Delta^2\right)\right]\left\{\frac{(u - t_T)^2}{t_T^2 (1 - t_T)^2}\right\} \\
&- \frac{b}{t_T(1 - t_T)} \mp \left\{\frac{2(2t_T - 1)}{t_T^2 (1 - t_T)^2}\right\}\Delta b + O\left(\Delta^2\right).
\end{aligned}$$

Then, it can be found via (S34)-(S36) that

$$I_{d_T}^{\pm} = \frac{d_T}{b^2}\left\{\pm \frac{\Delta b^{1/2}}{\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{3/2}} + o\left(\Delta b^{1/2}\right)\right\},$$

which implies that

$$I_{d_T}^{-} - I_{d_T}^{+} = -\sqrt{\frac{2}{\pi}}\frac{d_T}{\left\{t_T(1 - t_T)\right\}^{3/2}}\left(\frac{\Delta}{b^{3/2}}\right) + o\left(\frac{\Delta}{b^{3/2}}\right).$$

Hence, (S57) is demonstrated.

S.2.6.2 Proof of (S55)

By a mean-value expansion of $\hat{J}^{(2)}(\varsigma)$ around $\varsigma = t_T$,

$$\begin{aligned}
\hat{J}^{(2)}(\varsigma) - \hat{J}^{(2)}(t_T) &= \hat{J}^{(3)}(\bar{\varsigma})(\varsigma - t_T) \\
&= \hat{J}^{(3)}(t_T)(\varsigma - t_T) + \left\{\hat{J}^{(3)}(\bar{\varsigma}) - \hat{J}^{(3)}(t_T)\right\}(\varsigma - t_T)
\end{aligned}$$

for some $\bar{\varsigma}$ on the line segment joining ς and t_T. Then,

$$D_1 \leq \left(\frac{b^{3/2}}{\Delta}\right)\left\{\left|\hat{J}^{(3)}(t_T)\right| + \left|\hat{J}^{(3)}(\bar{\varsigma}) - \hat{J}^{(3)}(t_T)\right|\right\}|\varsigma - t_T|,$$

and it follows from Lipschitz continuity of $\hat{J}^{(3)}(\cdot)$ that $\left|\hat{J}^{(3)}(\bar{\varsigma}) - \hat{J}^{(3)}(t_T)\right|$ is of smaller order of magnitude than $\left|\hat{J}^{(3)}(t_T)\right|$. In short, $\left(b^{3/2}/\Delta\right)\left|\hat{J}^{(3)}(t_T)\right||\varsigma - t_T|$ is the dominant term in D_1.

In what follows, the order of magnitude in $\left|\hat{J}^{(3)}(t_T)\right|$ can be found via the identity $\hat{J}^{(3)}(t_T) \equiv E\left\{\hat{J}^{(3)}(t_T)\right\} + \left[\hat{J}^{(3)}(t_T) - E\left\{\hat{J}^{(3)}(t_T)\right\}\right]$. Now,

$$\begin{aligned}\left|E\left\{\hat{J}^{(3)}(t_T)\right\}\right| &\leq E\left|\dddot{K}^{-}_{t_T}(Y_i) - \dddot{K}^{+}_{t_T}(Y_i)\right| \\ &\leq \int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| g_Y(u)\,du + |d_T|\int_0^{t_T}\left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du,\end{aligned}$$

where

$$\begin{aligned}\dddot{K}^{\pm}_{t_T}(u) = \frac{1}{b^3}\Big[&\left\{\mathcal{L}^{\pm}_B(u)\right\}^3 - 3\left\{\Psi^{(1)}\left(p^{\pm}_{b,t_T,0}\right) + \Psi^{(1)}\left(q^{\pm}_{b,t_T}\right)\right\}\mathcal{L}^{\pm}_B(u) \\ &-\Psi^{(2)}\left(p^{\pm}_{b,t_T,0}\right) + \Psi^{(2)}\left(q^{\pm}_{b,t_T}\right)\Big] K^{\pm}_{t_T}(u).\end{aligned}$$

Because $g_Y(\cdot)$ is uniformly bounded on $[0,1]$ and $\int_0^{t_T}\left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du \leq \int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du$, we can see that the order of magnitude in $\left|E\left\{\hat{J}^{(3)}(t_T)\right\}\right|$ is determined by $\int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du$. A mean-value expansion of $\dddot{K}^{\pm}_{t_T}(u)$ around $\Delta = 0$ implies that

$$\begin{aligned}&\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u) \\ &\sim 2\left(\frac{\Delta}{b^4}\right)\Big[-\left\{\mathcal{L}^0_B(u)\right\}^4 + 6\left\{\Psi^{(1)}\left(p^0_{b,t_T,0}\right) + \Psi^{(1)}\left(q^0_{b,t_T}\right)\right\}\left\{\mathcal{L}^0_B(u)\right\}^2 \\ &+ 4\left\{\Psi^{(2)}\left(p^0_{b,t_T,0}\right) - \Psi^{(2)}\left(q^0_{b,t_T}\right)\right\}\mathcal{L}^0_B(u) \\ &-3\left\{\Psi^{(1)}\left(p^0_{b,t_T,0}\right) + \Psi^{(1)}\left(q^0_{b,t_T}\right)\right\}^2 + \left\{\Psi^{(3)}\left(p^0_{b,t_T,0}\right) + \Psi^{(3)}\left(q^0_{b,t_T}\right)\right\}\Big] K_{B(t_T,b)}(u),\end{aligned}$$

where $\mathcal{L}^0_B(u)$ is defined in (S28), and $K_{B(y,b)}(u)$ is Chen's (1999) original beta kernel. By extending Lemma A.2 of Funke and Hirukawa (2024) to higher-order moments of the log-transformed beta random variable, taking a similar approach to the proof of Lemma A6 and using (S7), it can be found that

$$\int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du \leq O\left(\frac{\Delta}{b^4}\right) O\left(b^2\right) = O\left(\frac{\Delta}{b^2}\right),$$

Hence, $\left|E\left\{\hat{J}^{(3)}(t_T)\right\}\right| \leq O\left(\Delta/b^2\right)$ is the case.

Furthermore,

$$Var\left\{\hat{J}^{(3)}(t_T)\right\} = \frac{1}{n}\left[E\left\{\dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i)\right\}^2 - E^2\left\{\dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i)\right\}\right],$$

where $\left|E\left\{\dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i)\right\}\right| = \left|E\left\{\hat{J}^{(3)}(t_T)\right\}\right| = O\left(\Delta/b^2\right)$ as above. By taking into account that the order of magnitude in $E\left\{\dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i)\right\}^2$ is determined by $\int_0^1 \left\{\dddot{K}_{t_T}^{-}(u) - \dddot{K}_{t_T}^{+}(u)\right\}^2 du$ and again following a similar procedure to the proof of Lemma A6, it can be shown that

$$\int_0^1 \left\{\dddot{K}_{t_T}^{-}(u) - \dddot{K}_{t_T}^{+}(u)\right\}^2 du \le O\left(\frac{\Delta^2}{b^8}\right) O\left(b^{-1/2}\right) O\left(b^4\right) = O\left(\frac{\Delta^2}{b^{9/2}}\right).$$

It follows that $Var\left\{\hat{J}^{(3)}(t_T)\right\} = O\left\{\Delta^2/\left(nb^{9/2}\right)\right\}$.

In conclusion,

$$\left|\hat{J}^{(3)}(t_T)\right| = O\left(\frac{\Delta}{b^2}\right) + O_p\left(\sqrt{\frac{\Delta^2}{nb^{9/2}}}\right).$$

Because ς lies between $\hat{t}_T$ and t_T and $\left|\hat{t}_T - t_T\right| = O_p(c_n) = o_p\left(b^{1/2}\right)$ as in the proof of Theorem 1, it holds that $|\varsigma - t_T| \le \left|\hat{t}_T - t_T\right| = o_p\left(b^{1/2}\right)$. Therefore,

$$\begin{aligned}\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(3)}(t_T)\right| |\varsigma - t_T| &= \left(\frac{b^{3/2}}{\Delta}\right)\left\{O\left(\frac{\Delta}{b^2}\right) + O_p\left(\sqrt{\frac{\Delta^2}{nb^{9/2}}}\right)\right\} o_p\left(b^{1/2}\right) \\ &= o_p(1) + o_p\left(\frac{1}{\sqrt{nb^{1/2}}}\right) \xrightarrow{p} 0,\end{aligned}$$

which establishes (S55).

S.2.6.3 Proof of (S56)

It suffices to show that $Var\left\{\left(b^{3/2}/\Delta\right)\hat{J}^{(2)}(t_T)\right\} = o(1)$. To do so, again we focus on the order of magnitude in $E\left\{\ddot{K}_{t_T}^{-}(Y_i) - \ddot{K}_{t_T}^{+}(Y_i)\right\}^2$. Taking a mean-value expansion of $\ddot{K}_{t_T}^{\pm}(u)$ around $\Delta = 0$ and using the same notation as in the proof of (S55) above lead to

$$\begin{aligned}&\ddot{K}_{t_T}^{-}(u) - \ddot{K}_{t_T}^{+}(u) \\ &\sim 2\left(\frac{\Delta}{b^3}\right)\left[\left\{\mathcal{L}_B^0(u)\right\}^3 - 3\left\{\Psi^{(1)}\left(p_{b,t_T,0}^0\right) + \Psi^{(1)}\left(q_{b,t_T}^0\right)\right\}\mathcal{L}_B^0(u)\right. \\ &\left.+\Psi^{(2)}\left(p_{b,t_T,0}^0\right) - \Psi^{(2)}\left(q_{b,t_T}^0\right)\right] K_{B(t_T,b)}(u).\end{aligned}$$

Then, a similar argument to the proof of (S55) establishes that

$$E\left\{\ddot{K}_{t_T}^{-}(Y_i) - \ddot{K}_{t_T}^{+}(Y_i)\right\}^2 = O\left(\frac{\Delta^2}{b^6}\right) O\left(b^{-1/2}\right) O\left(b^3\right) = O\left(\frac{\Delta^2}{b^{7/2}}\right).$$

Because $Var\left\{\hat{J}^{(2)}(t_T)\right\} = O\left\{\Delta^2/\left(nb^{7/2}\right)\right\}$, it holds that

$$Var\left\{\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}(t_T)\right\} = \left(\frac{b^3}{\Delta^2}\right)O\left(\frac{\Delta^2}{nb^{7/2}}\right) = O\left(\frac{1}{nb^{1/2}}\right) \to 0.$$

Therefore, (S56) is proven. This completes the proof. ■

S.2.7 Proof of Lemma A8

Basically, a similar strategy to the proofs of (S55) and (S56) in Lemma A7 may be taken. The derivation is straightforward but much more tedious, and thus details are omitted. ■

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