Supplement to "Nonparametric Estimation of Splicing Points in Actuarial Loss Distributions via Data Transformation"

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S1. Organization of the supplement

The remainder of this online supplement is organized as follows. Section S2 provides technical proofs of theorems in the main body. Propositions and lemmata that are required to establish the theorems are also presented with proofs. Furthermore, remarks on convergence results are made whenever necessary. Section S3 presents details of the Monte Carlo design and comprehensive simulation results. Section S4 shows descriptions of the datasets used in empirical applications and complete empirical results.

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S2. Additional convergence results and technical proofs

S2.1. Notation

This section adopts the following notational conventions: '$a_n \sim b_n$' means that a_n/b_n converges to 1; '$a_n = o(b_n)$' signifies that a_n/b_n converges to 0; '$a_n = O(b_n)$' means that a_n/b_n is bounded; and we say that '$a_n \asymp b_n$' if there exist constants $0 < c_1 < c_2 < \infty$ so that $c_1 a_n \leq b_n \leq c_2 a_n$. For a function $h(x)$ and a point c, $h(c^-) = \lim_{x \uparrow c} h(x)$, $h(c^+) = \lim_{x \downarrow c} h(x)$ and $h^{(m)}(x) = d^m h(x)/dx^m$ denote the left and right limits, and the mth-order derivative, respectively. For $a > 0$, $\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\, dt$ is the gamma function. $\Psi(a) = d \ln \Gamma(a)/da$ and $\Psi^{(m)}(a) = d^m \Psi(a)/da^m$ are the digamma and polygamma functions, respectively. The abbreviation '$a.s.$' stands for "almost surely". The expression '$X \stackrel{d}{=} Y$' reads "A random variable X obeys the distribution Y."

In addition, the following shorthand notation is employed whenever no confusion may arise: $\dot{K}_c^{\pm}(u) = \partial K_y^{\pm}(u)/\partial y|_{y=c}$; $\ddot{K}_c^{\pm}(u) = \partial^2 K_y^{\pm}(u)/\partial y^2|_{y=c}$; $\dddot{K}_c^{\pm}(u) = \partial^3 K_y^{\pm}(u)/\partial y^3|_{y=c}$; $H_i = \dot{K}_{t_T}^{-}(Y_i) - \dot{K}_{t_T}^{+}(Y_i)$. Finally, $Z_{b,y,m}^{\pm} \stackrel{d}{=} Beta(p_{b,y,m}^{\pm}, q_{b,y}^{\pm}) := Beta\{(y \pm \Delta)/b + m + 1, (1 - (y \pm \Delta))/b + 1\}$ for $m \in \{0, 1, 2, \ldots\}$, and $Z_{b,y,m}^{0} \stackrel{d}{=} Beta(p_{b,y,m}^{0}, q_{b,y}^{0}) := Beta\{y/b + m + 1, (1-y)/b + 1\}$.

S2.2. List of useful formulae

The formulae presented below are useful for the proofs. While (S2) is taken from Lemma A.2 of Funke and Hirukawa (2024), (S3) is a natural extension of the lemma and can be shown immediately. Moreover, (S4) is the same as equation (A3) of Funke and Hirukawa (2025), and (S5)-(S7) are taken from Lemma A1 of Funke and Hirukawa (2025) with "$x \in I_0$" replaced by "$y \in I_T$".

S2.2.1. Stirling's formula

As $a \to \infty$,

$$\Gamma(a+1) = \sqrt{2\pi} a^{a+1/2} \exp(-a) \left\{ 1 + \frac{1}{12a} + O\left(a^{-2}\right) \right\}. \tag{S1}$$

S2.2.2. Moments of the log-transformed beta random variable

Let $Z \overset{d}{=} Beta(\gamma, \delta)$. Then, for $m \in \{0, 1, 2, ...\}$,

$$E\left\{ Z^m \ln\left(\frac{Z}{1-Z}\right) \right\} = \frac{B(\gamma+m, \delta)}{B(\gamma, \delta)} \{\Psi(\gamma+m) - \Psi(\delta)\}, \tag{S2}$$

and

$$\begin{aligned} & E\left\{ Z^m \ln^2\left(\frac{Z}{1-Z}\right) \right\} \\ & = \frac{B(\gamma+m, \delta)}{B(\gamma, \delta)} \left[\{\Psi(\gamma+m) - \Psi(\delta)\}^2 + \Psi^{(1)}(\gamma+m) + \Psi^{(1)}(\delta) \right], \end{aligned} \tag{S3}$$

where

$$\frac{B(\gamma+m, \delta)}{B(\gamma, \delta)} = \begin{cases} 1 & \text{for } m = 0 \\ \dfrac{\prod_{k=1}^{m}(\gamma+k-1)}{\prod_{k=1}^{m}(\gamma+\delta+k-1)} & \text{for } m \geq 1 \end{cases}.$$

S2.2.3. Recursive formulae for digamma and trigamma functions

For $a > 0$ and $m \in \{0, 1, 2, ...\}$,

$$\Psi^{(m)}(a+1) = \Psi^{(m)}(a) + \frac{(-1)^m \, m!}{a^{m+1}}. \tag{S4}$$

S2.2.4. Approximations to digamma and trigamma functions

As $b \to 0$,

$$\sup_{y \in I_T} \left| \Psi\left(\frac{y}{b}+1\right) - \left\{ \ln\left(\frac{y}{b}\right) + \frac{b}{2y} \right\} \right| = O\left(b^2\right), \tag{S5}$$

$$\sup_{y \in I_T} \left| \Psi^{(1)}\left(\frac{y}{b}+1\right) - \left(\frac{b}{y} - \frac{b^2}{2y^2} \right) \right| = O\left(b^3\right), \tag{S6}$$

and for $m \geq 2$,

$$\sup_{y \in I_T} \left| \Psi^{(m)}\left(\frac{y}{b}+1\right) \right| = O\left(b^m\right). \tag{S7}$$

S2.3. Propositions about approximating the diagnostics

Two propositions about the properties of diagnostics are presented below. Proposition S1 demonstrates uniform bounds of density estimators $\hat{f}_Y^{\pm}(y)$ on I_T, whereas Proposition S2 establishes that maximizing $|\hat{J}(y)|$ on I_T is a well-defined problem.

Before presenting the propositions, we make a few remarks on Assumption 4. This assumption controls the shrinkage rates of tuning parameters b and Δ. More specifically, it leads to: (i) $b = o(\Delta)$; (ii) $b^{1/2} = o(\Delta^2/b)$; and (iii) $\Delta = o(\Delta^3/b^{3/2})$. These are frequently used to control remainder terms in asymptotic expansions. It follows from $b = o(\Delta)$ and $\Delta = o(b^{1/2})$ that Δ

should shrink more slowly than b but faster than $b^{1/2}$. Assumption 4 also implies that $\ln n/(nb^{1/2}) \to 0$, which is a prerequisite for Proposition S1 below.

S2.3.1. Proposition about uniform approximations to density estimators

Proposition S1. *If Assumptions 1-4 hold, then*

$$\sup_{y \in I_T} \left| E\{\hat{f}_Y^{\pm}(y)\} - \left\{ g_Y(y) \pm g_Y^{(1)}(y)\Delta + d_T \int_0^{t_T} K_y^{\pm}(u)\, du \right\} \right| = O(b), \quad \text{(S8)}$$

and

$$\sup_{y \in I_T} \left| \hat{f}_Y^{\pm}(y) - E\{\hat{f}_Y^{\pm}(y)\} \right| = O\left(\sqrt{\frac{\ln n}{nb^{1/2}}} \right) \ a.s., \quad \text{(S9)}$$

as $n \to \infty$.

A direct outcome from this proposition is that $|E\{\hat{J}(y)\}|$ constitutes the dominant term in $|\hat{J}(y)|$ in a first-order asymptotic sense:

$$|\hat{J}(y)| = |E\{\hat{J}(y)\}| + O\left(\sqrt{\frac{\ln n}{nb^{1/2}}} \right) \ a.s.$$

uniformly on I_T. Specifically, by (S8), $|E\{\hat{J}(y)\}| := |d_T|\, J(y) + O(\Delta)$, where

$$\begin{aligned} J(y) &:= \int_0^{t_T} K_y^{-}(u)\, du - \int_0^{t_T} K_y^{+}(u)\, du \\ &= B\left(\frac{y-\Delta}{b} + 1, \frac{1-y+\Delta}{b} + 1; t_T \right) \\ &- B\left(\frac{y+\Delta}{b} + 1, \frac{1-y-\Delta}{b} + 1; t_T \right), \end{aligned}$$

and

$$B(p,q;r) = \frac{\int_0^r y^{p-1}(1-y)^{q-1}\,dy}{B(p,q)}$$

for $p, q > 0$, $r \in [0,1]$ is the incomplete beta function ratio.

S2.3.2. Proposition about the maximization problem

It is natural to verify whether $J(y)$ on $y \in I_T$ indeed has a unique maximum at t_T (or within a shrinking neighborhood of t_T even if it is not maximized exactly at this point). In reality, however, it is quite cumbersome to look into the local property of $J(y)$ analytically. It is well known that as $p, q \to \infty$, the probability density function (pdf) of $Beta\,(p,q)$ can be approximated by a normal pdf. Lemma S1 in Section S2.5 formally offers such an approximation, which is a refinement of Lemma A.1 of Moscovich et al. (2016). Based on Lemma S1, the next proposition refers to properties of the approximation and the maximizer of the approximated function.

Proposition S2. *If Assumption 4 holds, then the following hold true.*

(i) *Define*

$$Q(y) := \left[\frac{(1+2b)\sqrt{1+3b}\{(1-2y)t_T+(y+b)\}}{\{(y+b)(1-y+b)\}^{3/2}}\right] \\ \times \phi\left\{\sqrt{\frac{1+3b}{b(y+b)(1-y+b)}}(t_T-y+(2t_T-1)b)\right\},$$

where $\phi(\cdot)$ denotes the pdf of $N(0,1)$. Then,

$$\sup_{y\in I_T}\left|J\left(y\right)-Q\left(y\right)\left(\frac{\Delta}{b^{1/2}}\right)\right|=O\left(\frac{\Delta^3}{b^{3/2}}\right),$$

as $n\to\infty$.

(ii) *$Q(y)$ on I_T has a unique maximum at $y=t_T^*=t_T+2(2t_T-1)b+O(b^2)$, as $n\to\infty$.*

As a consequence of Propositions S1 and S2, it holds that

$$\left|\hat{J}\left(y\right)\right|\sim\left|d_T\right|J\left(y\right)\sim\left|d_T\right|Q\left(y\right)\left(\frac{\Delta}{b^{1/2}}\right)$$

on $y\in I_T$. It follows that maximizing the diagnostic function $|\hat{J}(y)|$ on $y\in I_T$ is a well-defined problem. Furthermore, Proposition S2(ii) suggests that $t_T^*=t_T+O\left(b^2\right)$ when $t_T=1/2$. In this case, the maximizer of $Q\left(y\right)$ is considerably close to the true splicing point in the transformed scale, as are the maximizers of $J\left(y\right)$ and $\left|\hat{J}\left(y\right)\right|$. Although the location of the splicing point in the original scale is unknown, we have a high chance to estimate it precisely in the transformed scale as long as it is mapped to (a neighborhood of) 1/2. This is a rationale of mapping the midpoint of I_0 to 1/2 in Assumption 3(i).

S2.4. Proof of Proposition S1

This proposition can be established by a minor modification of the proof of Theorem 2 in Hirukawa et al. (2022), and thus details are omitted. ■

S2.5. Proof of Proposition S2

The proof requires the following lemmata. We recommend that interested readers verify the calculations in all proofs hereinafter with the aid of Maple™ or Mathematica®.

Lemma S1. *Let* $Z \stackrel{d}{=} Beta\left(p, q\right)$*, where* $p, q \to \infty$ *and* $p \asymp q$ *so that both* $q = O\left(p\right)$ *and* $p = O\left(q\right)$ *hold. Also suppose that the argument* z *in the pdf of the beta random variable* Z *admits the location-scale transformation* $z := \mu + \sigma v$*, where*

$$\mu = E\left(Z\right) = \frac{p}{p+q},$$

and

$$\sigma^2 = Var\left(Z\right) = \frac{pq}{\left(p+q\right)^2 \left(p+q+1\right)}.$$

Then, the pdf of Z *can be approximated by*

$$\begin{aligned} f_Z\left(z\right) &= f_Z\left(\mu + \sigma v\right) \\ &= \frac{\phi\left(v\right)}{\sigma} \left[1 + \frac{v}{\sqrt{p+q+1}} \left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)\right. \\ &\left. + \frac{v^3}{3} \left\{ \left(\frac{q}{p+q}\right)^{3/2} \frac{1}{\sqrt{p}} - \left(\frac{p}{p+q}\right)^{3/2} \frac{1}{\sqrt{q}} \right\} + O\left(p^{-1}\right) \right]. \end{aligned}$$

Lemma S2. *Let* $f_{Z^{\pm}_{b,y,m}}\left(z\right)$, $\mu^{\pm}_{b,y,m}$ and $\left(\sigma^{\pm}_{b,y,m}\right)^2$ *be the pdf, the mean and the variance of the beta random variable* $Z^{\pm}_{b,y,m}$*, respectively, where*

$$\mu^{\pm}_{b,y,m} = E\left(Z^{\pm}_{b,y,m}\right) = \frac{y \pm \Delta + \left(m+1\right)b}{1 + \left(m+2\right)b}$$

and

$$\left(\sigma_{b,y,m}^{\pm}\right)^2 = Var\left(Z_{b,y,m}^{\pm}\right) = \frac{b\left\{y \pm \Delta + (m+1)\,b\right\}\left\{1-(y\pm\Delta)+b\right\}}{\left\{1+(m+2)\,b\right\}^2\left\{1+(m+3)\,b\right\}}.$$

Then, as $n \to \infty$, the incomplete beta function ratio $B\left(p_{b,y,m}^{\pm}, q_{b,y}^{\pm}; t_T\right) = \int_0^{t_T} f_{Z_{b,y,m}^{\pm}}(z)\,dz$ can be approximated by

$$\begin{aligned}&\int_0^{t_T} f_{Z_{b,y,m}^{\pm}}(z)\,dz\\ &:= C_{b,y,m}^{\pm}(0) + b^{1/2}\left\{\xi_{b,y,m}^{\pm} C_{b,y,m}^{\pm}(1) + \zeta_{b,y,m}^{\pm} C_{b,y,m}^{\pm}(3)\right\} + O(b)\,,\end{aligned} \tag{S10}$$

where

$$\begin{aligned}
C_{b,y,m}^{\pm}(k) &= \int_{B_{b,y,m}^{\pm}}^{A_{b,y,m}^{\pm}} v^k \phi(v)\,dv \text{ for } k=0,1,3,\\
A_{b,y,m}^{\pm} &= \frac{t_T - \mu_{b,y,m}^{\pm}}{\sigma_{b,y,m}^{\pm}} = \frac{\left[t_T - y \mp \Delta + \left\{(m+2)\,t_T - (m+1)\right\} b\right]\sqrt{1+(m+3)\,b}}{\sqrt{b\left\{(y\pm\Delta)+(m+1)\,b\right\}\left\{1-(y\pm\Delta)+b\right\}}},\\
B_{b,y,m}^{\pm} &= -\frac{\mu_{b,y,m}^{\pm}}{\sigma_{b,y,m}^{\pm}} = \frac{\left\{-y \mp \Delta - (m+1)\,b\right\}\sqrt{1+(m+3)\,b}}{\sqrt{b\left\{(y\pm\Delta)+(m+1)\,b\right\}\left\{1-(y\pm\Delta)+b\right\}}},\\
\xi_{b,y,m}^{\pm} &= \frac{b^{-1/2}}{\sqrt{p_{b,y,m}^{\pm} + q_{b,y}^{\pm} + 1}}\left(\sqrt{\frac{p_{b,y,m}^{\pm}}{q_{b,y}^{\pm}}} - \sqrt{\frac{q_{b,y}^{\pm}}{p_{b,y,m}^{\pm}}}\right),\\
\zeta_{b,y,m}^{\pm} &= \frac{b^{-1/2}}{3}\left\{\left(\frac{q_{b,y}^{\pm}}{p_{b,y,m}^{\pm}+q_{b,y}^{\pm}}\right)^{3/2}\frac{1}{\sqrt{p_{b,y,m}^{\pm}}} - \left(\frac{p_{b,y,m}^{\pm}}{p_{b,y,m}^{\pm}+q_{b,y}^{\pm}}\right)^{3/2}\frac{1}{\sqrt{q_{b,y}^{\pm}}}\right\},
\end{aligned}$$

and the $O(b)$ rate in (S10) is uniform on $y \in I_T$. Furthermore, as $n \to \infty$,

$$\xi^{\pm}_{b,y,m} = \frac{2y-1}{\sqrt{y(1-y)}} \pm \frac{\Delta}{2\{y(1-y)\}^{3/2}} + O(b),$$

and

$$\zeta^{\pm}_{b,y,m} = \frac{1-2y}{3\sqrt{y(1-y)}} \mp \frac{\Delta}{6\{y(1-y)\}^{3/2}} + O(b),$$

where $O(b)$ rates are again uniform on $y \in I_T$.

Lemma S3 (Komatsu, 1955)**.** *For $x > 0$,*

$$\frac{2}{\sqrt{x^2+4}+x} < e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt < \frac{2}{\sqrt{x^2+2}+x}.$$

S2.5.1. Proof of Lemma S1

The proof closely follows the one of Lemma A.1 in Moscovich et al. (2016). The beta random variable $Z \overset{d}{=} Beta(p,q)$ has the density

$$f_Z(z) = \frac{z^{p-1}(1-z)^{q-1}}{B(p,q)} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} z^{p-1}(1-z)^{q-1}.$$

By the change of variables $z := \mu + \sigma v$ with

$$(\mu, \sigma^2) = \left(\frac{p}{p+q}, \frac{pq}{(p+q)^2(p+q+1)}\right),$$

the density can be rearranged to

$$\begin{aligned}
f_Z(z) &= f_Z(\mu+\sigma v) \\
&= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}(\mu+\sigma v)^{p-1}(1-\mu-\sigma v)^{q-1} \\
&= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\mu^{p-1}(1-\mu)^{q-1}\left(1+\frac{\sigma}{\mu}v\right)^{p-1}\left(1-\frac{\sigma}{1-\mu}v\right)^{q-1}.
\end{aligned}$$

Then, the proof boils down to demonstrating that

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\mu^{p-1}(1-\mu)^{q-1} = \frac{1}{\sqrt{2\pi}\sigma}\left\{1+O\left(p^{-1}\right)\right\}, \tag{S11}$$

and that

$$\begin{aligned}
&\left(1+\frac{\sigma}{\mu}v\right)^{p-1}\left(1-\frac{\sigma}{1-\mu}v\right)^{q-1} \\
&= \exp\left(-\frac{v^2}{2}\right)\left[1+\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right. \\
&\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right],
\end{aligned} \tag{S12}$$

as $p, q \to \infty$ and $p \asymp q$.

Proof of (S11). By the definition of μ,

$$\mu^{p-1}(1-\mu)^{q-1} = \frac{p^{p-1}q^{q-1}}{(p+q)^{p+q-2}}.$$

In addition, by the property of the gamma function,

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} = \left(\frac{pq}{p+q}\right)\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)}.$$

Applying (S1) to three gamma functions under $p, q \to \infty$ and $p \asymp q$ gives

$$\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} = \frac{(p+q)^{p+q+1/2}}{\sqrt{2\pi}p^{p+1/2}q^{q+1/2}}\left\{1+O\left(p^{-1}\right)\right\}.$$

Therefore,

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\mu^{p-1}(1-\mu)^{q-1} = \frac{(p+q)^{3/2}}{\sqrt{2\pi}\sqrt{pq}}\left\{1+O\left(p^{-1}\right)\right\}.$$

Observe that the right-hand side can be rewritten as

$$\begin{aligned}&\frac{(p+q)\sqrt{(p+q+1)-1}}{\sqrt{2\pi}\sqrt{pq}}\left\{1+O\left(p^{-1}\right)\right\}\\ &= \frac{(p+q)\sqrt{p+q+1}}{\sqrt{2\pi}\sqrt{pq}}\sqrt{1-\frac{1}{p+q+1}}\left\{1+O\left(p^{-1}\right)\right\}.\end{aligned}$$

Then, (S11) can be demonstrated by recognizing that

$$\frac{1}{\sigma} = \frac{(p+q)\sqrt{p+q+1}}{\sqrt{pq}}$$

and that

$$\sqrt{1-\frac{1}{p+q+1}} = 1+O\left(p^{-1}\right).$$

Proof of (S12). For

$$\begin{aligned}
&\left(1+\frac{\sigma}{\mu}v\right)^{p-1}\left(1-\frac{\sigma}{1-\mu}v\right)^{q-1} \\
&= \exp\left\{(p-1)\ln\left(1+\frac{\sigma}{\mu}v\right)+(q-1)\ln\left(1-\frac{\sigma}{1-\mu}v\right)\right\},
\end{aligned}$$

observe that

$$\left(\frac{\sigma}{\mu},\frac{\sigma}{1-\mu}\right)=\left(\sqrt{\frac{q}{p\left(p+q+1\right)}},\sqrt{\frac{p}{q\left(p+q+1\right)}}\right)$$

are both $O\left(p^{-1/2}\right)\to 0$. Then, by a fourth-order Taylor expansion,

$$\begin{aligned}
&\ln\left(1+\frac{\sigma}{\mu}v\right) \\
&= \left(\frac{\sigma}{\mu}\right)v-\frac{1}{2}\left(\frac{\sigma}{\mu}\right)^2 v^2+\frac{1}{3}\left(\frac{\sigma}{\mu}\right)^3 v^3+O\left(p^{-2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\ln\left(1-\frac{\sigma}{1-\mu}v\right) \\
&= -\left(\frac{\sigma}{1-\mu}\right)v-\frac{1}{2}\left(\frac{\sigma}{1-\mu}\right)^2 v^2-\frac{1}{3}\left(\frac{\sigma}{1-\mu}\right)^3 v^3+O\left(p^{-2}\right).
\end{aligned}$$

Hence,

$$(p-1)\ln\left(1+\frac{\sigma}{\mu}v\right)+(q-1)\ln\left(1-\frac{\sigma}{1-\mu}v\right)$$
$$=\sigma\left(\frac{p-1}{\mu}-\frac{q-1}{1-\mu}\right)v-\frac{\sigma^2}{2}\left\{\frac{p-1}{\mu^2}+\frac{q-1}{(1-\mu)^2}\right\}v^2$$
$$+\frac{\sigma^3}{3}\left\{\frac{p-1}{\mu^3}-\frac{q-1}{(1-\mu)^3}\right\}v^3+O\left(p^{-1}\right). \tag{S13}$$

Again by the definitions of (μ,σ^2), $p,q\to\infty$ and $p\asymp q$,

$$\sigma\left(\frac{p-1}{\mu}-\frac{q-1}{1-\mu}\right)=\frac{1}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right), \tag{S14}$$

and

$$\begin{aligned}\sigma^2\left\{\frac{p-1}{\mu^2}+\frac{q-1}{(1-\mu)^2}\right\}&=\frac{pq}{p+q+1}\left(\frac{p-1}{p^2}+\frac{q-1}{q^2}\right)\\&=\frac{pq}{p+q+1}\left(\frac{1}{p}+\frac{1}{q}\right)\left\{1+O\left(p^{-1}\right)\right\}\\&=\frac{p+q}{p+q+1}\left\{1+O\left(p^{-1}\right)\right\}\\&=1+O\left(p^{-1}\right).\end{aligned} \tag{S15}$$

Moreover,

$$\sigma^3\left\{\frac{p-1}{\mu^3}-\frac{q-1}{(1-\mu)^3}\right\}$$
$$=\left(\frac{q}{p+q+1}\right)^{3/2}\left(\frac{p-1}{p^{3/2}}\right)-\left(\frac{p}{p+q+1}\right)^{3/2}\left(\frac{q-1}{q^{3/2}}\right),$$

where

$$\frac{p}{p+q+1} = \frac{p}{p+q}\left\{1+O\left(p^{-1}\right)\right\},$$
$$\frac{p-1}{p^{3/2}} = \frac{1}{\sqrt{p}}\left\{1+O\left(p^{-1}\right)\right\},$$

and so forth. It follows that

$$\begin{aligned}
&\sigma^3\left\{\frac{p-1}{\mu^3} - \frac{q-1}{(1-\mu)^3}\right\} \\
&= \left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}} - \left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}\left\{1+O\left(p^{-1}\right)\right\}. \qquad \text{(S16)}
\end{aligned}$$

Therefore, by (S13)-(S16),

$$\begin{aligned}
&\left(1+\frac{\sigma}{\mu}v\right)^{p-1}\left(1-\frac{\sigma}{1-\mu}v\right)^{q-1} \\
&= \exp\left\{\sigma\left(\frac{p-1}{\mu} - \frac{q-1}{1-\mu}\right)v\right\}\exp\left\{-\frac{\sigma^2}{2}\left\{\frac{p-1}{\mu^2} + \frac{q-1}{(1-\mu)^2}\right\}v^2\right\} \\
&\times \exp\left\{\frac{\sigma^3}{3}\left\{\frac{p-1}{\mu^3} - \frac{q-1}{(1-\mu)^3}\right\}v^3\right\}\exp\left\{O\left(p^{-1}\right)\right\}
\end{aligned}$$

$$= \exp\left\{\underbrace{\frac{1}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)v}_{=O\left(p^{-1/2}\right)}\right\}\underbrace{\exp\left[-\frac{v^2}{2}\left\{1+O\left(p^{-1}\right)\right\}\right]}_{=\exp\left(-\frac{v^2}{2}\right)\exp\{O(p^{-1})\}}$$

$$\times \exp\left[\frac{1}{3}\left\{\underbrace{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}}_{=O\left(p^{-1/2}\right)}\right\}\left\{1+O\left(p^{-1}\right)\right\}v^3\right]\underbrace{\exp\left\{O\left(p^{-1}\right)\right\}}_{=1+O(p^{-1})}$$

$$= \exp\left(-\frac{v^2}{2}\right)\exp\left[\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right.$$

$$\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right]\left\{1+O\left(p^{-1}\right)\right\}$$

$$= \exp\left(-\frac{v^2}{2}\right)\left[1+\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right.$$

$$\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right]\left\{1+O\left(p^{-1}\right)\right\}$$

$$= \exp\left(-\frac{v^2}{2}\right)\left[1+\frac{v}{\sqrt{p+q+1}}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)\right.$$

$$\left.+\frac{v^3}{3}\left\{\left(\frac{q}{p+q}\right)^{3/2}\frac{1}{\sqrt{p}}-\left(\frac{p}{p+q}\right)^{3/2}\frac{1}{\sqrt{q}}\right\}+O\left(p^{-1}\right)\right],$$

and thus (S12) is established. This completes the proof. ■

S2.5.2. Proof of Lemma S2

Because (S10) is a direct outcome from Lemma S1, it suffices to derive approximations to $\xi^{\pm}_{b,y,m}$ and $\zeta^{\pm}_{b,y,m}$. These are built on the following results as $n \to \infty$, where the $O(b)$ rate in each approximation is uniform on $y \in I_T$:

$$\sqrt{\frac{p^{\pm}_{b,y,m}}{q^{\pm}_{b,y}}} = \sqrt{\frac{y}{1-y}} \left\{ 1 \pm \frac{\Delta}{2y(1-y)} + O(b) \right\}. \tag{S17}$$

$$\sqrt{\frac{q^{\pm}_{b,y}}{p^{\pm}_{b,y,m}}} = \sqrt{\frac{1-y}{y}} \left\{ 1 \mp \frac{\Delta}{2y(1-y)} + O(b) \right\}. \tag{S18}$$

$$\frac{1}{\sqrt{p^{\pm}_{b,y,m} + q^{\pm}_{b,y} + 1}} = b^{1/2} \{1 + O(b)\}. \tag{S19}$$

$$\left(\frac{p^{\pm}_{b,y,m}}{p^{\pm}_{b,y,m} + q^{\pm}_{b,y}} \right)^{3/2} = \sqrt{y} \left\{ y \pm \frac{3}{2}\Delta + O(b) \right\}. \tag{S20}$$

$$\left(\frac{q^{\pm}_{b,y}}{p^{\pm}_{b,y,m} + q^{\pm}_{b,y}} \right)^{3/2} = \sqrt{1-y} \left\{ (1-y) \mp \frac{3}{2}\Delta + O(b) \right\}. \tag{S21}$$

$$\frac{1}{\sqrt{p^{\pm}_{b,y,m}}} = \frac{b^{1/2}}{\sqrt{y}} \left\{ 1 \mp \frac{\Delta}{2y} + O(b) \right\}. \tag{S22}$$

$$\frac{1}{\sqrt{q^{\pm}_{b,y}}} = \frac{b^{1/2}}{\sqrt{1-y}} \left\{ 1 \pm \frac{\Delta}{2(1-y)} + O(b) \right\}. \tag{S23}$$

These approximations can be obtained in a similar manner, and thus we only present the derivation of (S17). By a second-order Taylor expansion of $\sqrt{p^{\pm}_{b,y,m}/q^{\pm}_{b,y}}$ around $\Delta = 0$ and $\Delta^2 = o(b)$,

$$\begin{aligned}
\sqrt{\frac{p^{\pm}_{b,y,m}}{q^{\pm}_{b,y}}} &= \sqrt{\frac{y+(m+1)b}{1-y+b}} \pm \frac{1+(m+2)b}{2(1-y+b)^{3/2}\sqrt{y+(m+1)b}}\Delta + O\left(\Delta^2\right) \\
&= \sqrt{\frac{y}{1-y}} \{1 + O(b)\} \pm \frac{1}{2(1-y)^{3/2}\sqrt{y}} \{1 + O(b)\} \Delta + o(b) \\
&= \sqrt{\frac{y}{1-y}} \left\{ 1 \pm \frac{\Delta}{2y(1-y)} + O(b) \right\}.
\end{aligned}$$

Substituting (S17)-(S23) into the definition of $\xi^{\pm}_{b,y,m}$ yields

$$
\begin{aligned}
&\xi^{\pm}_{b,y,m} \\
&= \frac{b^{-1/2}}{\sqrt{p^{\pm}_{b,y,m} + q^{\pm}_{b,y} + 1}} \left(\sqrt{\frac{p^{\pm}_{b,y,m}}{q^{\pm}_{b,y}}} - \sqrt{\frac{q^{\pm}_{b,y}}{p^{\pm}_{b,y,m}}} \right) \\
&= b^{-1/2} b^{1/2} \left\{1 + O\left(b\right)\right\} \left[\sqrt{\frac{y}{1-y}} \left\{ 1 \pm \frac{\Delta}{2y\left(1-y\right)} + O\left(b\right) \right\} \right. \\
&\left. - \sqrt{\frac{1-y}{y}} \left\{ 1 \mp \frac{\Delta}{2y\left(1-y\right)} + O\left(b\right) \right\} \right] \\
&= \frac{\left\{1 + O\left(b\right)\right\}}{\sqrt{y\left(1-y\right)}} \left[y \left\{ 1 \pm \frac{\Delta}{2y\left(1-y\right)} + O\left(b\right) \right\} - \left(1-y\right) \left\{ 1 \mp \frac{\Delta}{2y\left(1-y\right)} + O\left(b\right) \right\} \right] \\
&= \frac{2y-1}{\sqrt{y\left(1-y\right)}} \pm \frac{\Delta}{2\left\{y\left(1-y\right)\right\}^{3/2}} + O\left(b\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\zeta^{\pm}_{b,y,m} \\
&= \frac{b^{-1/2}}{3} \left\{ \left(\frac{q^{\pm}_{b,y}}{p^{\pm}_{b,y,m} + q^{\pm}_{b,y}} \right)^{3/2} \frac{1}{\sqrt{p^{\pm}_{b,y,m}}} - \left(\frac{p^{\pm}_{b,y,m}}{p^{\pm}_{b,y,m} + q^{\pm}_{b,y}} \right)^{3/2} \frac{1}{\sqrt{q^{\pm}_{b,y}}} \right\} \\
&= \frac{b^{-1/2}}{3} \left[\sqrt{1-y} \left\{ \left(1-y\right) \mp \frac{3}{2}\Delta + O\left(b\right) \right\} \frac{b^{1/2}}{\sqrt{y}} \left\{ 1 \mp \frac{\Delta}{2y} + O\left(b\right) \right\} \right. \\
&\left. - \sqrt{y} \left\{ y \pm \frac{3}{2}\Delta + O\left(b\right) \right\} \frac{b^{1/2}}{\sqrt{1-y}} \left\{ 1 \pm \frac{\Delta}{2\left(1-y\right)} + O\left(b\right) \right\} \right] \\
&= \frac{1}{3\sqrt{y\left(1-y\right)}} \left[\left(1-y\right) \left\{ \left(1-y\right) \mp \frac{1+2y}{2y}\Delta + O\left(b\right) \right\} - y \left\{ y \pm \frac{3-2y}{2\left(1-y\right)}\Delta + O\left(b\right) \right\} \right] \\
&= \frac{1-2y}{3\sqrt{y\left(1-y\right)}} \mp \frac{\Delta}{6\left\{y\left(1-y\right)\right\}^{3/2}} + O\left(b\right).
\end{aligned}
$$

The proof is completed by observing that the $O(b)$ rate in each approximation is uniform on $y \in I_T$. ■

S2.5.3. Proof of Proposition S2

Proof of (i). Put $m = 0$ in (S10). In this case, $\int_0^{t_T} K_y^{\pm}(u)\,du = B\left(p_{b,y,0}^{\pm}, q_{b,y}^{\pm}; t_T\right) = \int_0^{t_T} f_{Z_{b,y,0}^{\pm}}(z)\,dz$. Lemma S2 implies that each of $\xi_{b,y,0}^{\pm}$, $\zeta_{b,y,0}^{\pm}$, $C_{b,y,0}^{\pm}(1)$, and $C_{b,y,0}^{\pm}(3)$ is at most $O(1)$ uniformly on $y \in I_T$. Then,

$$\int_0^{t_T} K_y^{\pm}(u)\,du = C_{b,y,0}^{\pm}(0) + O\left(b^{1/2}\right) = \Phi\left(A_{b,y,0}^{\pm}\right) - \Phi\left(B_{b,y,0}^{\pm}\right) + O\left(b^{1/2}\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of $N(0,1)$ and the $O\left(b^{1/2}\right)$ rate is uniform on $y \in I_T$.

We start from working on $\Phi\left(A_{b,y,0}^{\pm}\right)$, where

$$A_{b,y,0}^{\pm} = \frac{\{t_T - y \mp \Delta + (2t_T - 1)\,b\}\sqrt{1+3b}}{\sqrt{b\{(y \pm \Delta) + b\}\{1 - (y \pm \Delta) + b\}}}.$$

By a third-order Taylor expansion of $\Phi\left(A_{b,y,0}^{\pm}\right)$ around $\Delta = 0$,

$$\Phi\left(A_{b,y,0}^{\pm}\right) = \Phi\left(A_{b,y,0}^{\pm}\right)\Big|_{\Delta=0} + \phi\left(A_{b,y,0}^{\pm}\right)\left.\frac{\partial A_{b,y,0}^{\pm}}{\partial \Delta}\right|_{\Delta=0}\Delta + O\left(\frac{\Delta^2}{b}\right) + O\left(\frac{\Delta^3}{b^{3/2}}\right),$$

where the coefficient on the $O\left(\Delta^2/b\right)$ term is shown to be common across $\Phi\left(A_{b,y,0}^{+}\right)$ and $\Phi\left(A_{b,y,0}^{-}\right)$ (although it is not specified explicitly), and the

$O\left(\Delta^3/b^{3/2}\right)$ rate is uniform on $y \in I_T$. By straightforward calculations,

$$A^{\pm}_{b,y,0}\big|_{\Delta=\ 0} = \sqrt{\frac{1+3b}{b\,(y+b)\,(1-y+b)}}\,\{t_T - y + (2t_T - 1)\,b\} := A_{b,y,0},$$

and

$$\left.\frac{\partial A^{\pm}_{b,y,0}}{\partial \Delta}\right|_{\Delta=0} = \mp\frac{(1+2b)\sqrt{1+3b}\,\{(1-2y)\,t_T + (y+b)\}}{2b^{1/2}\,\{(y+b)\,(1-y+b)\}^{3/2}}.$$

It follows that

$$\Phi\left(A^{\pm}_{b,y,0}\right) = \Phi\left(A_{b,y,0}\right) \mp \phi\left(A_{b,y,0}\right)\frac{(1+2b)\sqrt{1+3b}\,\{(1-2y)\,t_T + (y+b)\}}{2b^{1/2}\,\{(y+b)\,(1-y+b)\}^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right)$$
$$+O\left(\frac{\Delta^2}{b}\right) + O\left(\frac{\Delta^3}{b^{3/2}}\right).$$

Next, we demonstrate that for each fixed m, $\Phi\left(B^{\pm}_{b,y,m}\right) \to 0$ at an exponential rate as $n \to \infty$. It can be immediately found that $B^{\pm}_{b,y,m} = -b^{-1/2}\sqrt{y/(1-y)}\,\{1+o\,(1)\}$ for each fixed m, where the $o\,(1)$ term is uniform on $y \in I_T$. Hence, on $y \in I_T$, there are constants $0 < \underline{C}_B < \overline{C}_B < \infty$ so that $\underline{C}_B b^{-1/2} \le \left|B^{\pm}_{b,y,m}\right| \le \overline{C}_B b^{-1/2}$. In addition, Lemma S3 implies that

$$\frac{2e^{-x^2/2}}{\sqrt{2\pi}\left(\sqrt{x^2+4}+x\right)} < \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt < \frac{2e^{-x^2/2}}{\sqrt{2\pi}\left(\sqrt{x^2+2}+x\right)}.$$

When $x = \left|B^{\pm}_{b,y,m}\right|$, lower and upper bounds of this double inequality are bounded respectively by

$$\frac{2e^{-x^2/2}}{\sqrt{2\pi}\left(\sqrt{x^2+4}+x\right)} \geq \frac{2e^{-\overline{C}_B^2/(2b)}}{\sqrt{2\pi}\left(\sqrt{\overline{C}_B^2/b+4}+\overline{C}_B/b^{1/2}\right)} = O\left\{b^{1/2}\exp\left(-\frac{\overline{C}_B^2}{2b}\right)\right\},$$

and

$$\frac{2e^{-x^2/2}}{\sqrt{2\pi}\left(\sqrt{x^2+2}+x\right)} \leq \frac{2e^{-\underline{C}_B^2/(2b)}}{\sqrt{2\pi}\left(\sqrt{\underline{C}_B^2/b+2}+\underline{C}_B/b^{1/2}\right)} = O\left\{b^{1/2}\exp\left(-\frac{\underline{C}_B^2}{2b}\right)\right\}.$$

These converge to zero at an exponential rate as $n \to \infty$, so does $\Phi\left(B_{b,y,m}^{\pm}\right) = \int_{\left|B_{b,y,m}^{\pm}\right|}^{\infty}\left(e^{-t^2/2}/\sqrt{2\pi}\right)dt$.

Combining above results, we conclude that

$$\begin{aligned}
&\int_0^{t_T} K_y^{\pm}(u)\,du \\
&= \Phi\left(A_{b,y,0}\right) \mp \phi\left(A_{b,y,0}\right)\frac{(1+2b)\sqrt{1+3b}\left\{(1-2y)\,t_T+(y+b)\right\}}{2b^{1/2}\left\{(y+b)(1-y+b)\right\}^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right) \\
&+ O\left(\frac{\Delta^2}{b}\right) + O\left(\frac{\Delta^3}{b^{3/2}}\right).
\end{aligned}$$

Part (i) can be established because $J(y) = \int_0^{t_T} K_y^{-}(u)\,du - \int_0^{t_T} K_y^{+}(u)\,du$ holds, the $O\left(\Delta/b^{1/2}\right)$ terms can be rewritten as $\mp(1/2)\,Q(y)\left(\Delta/b^{1/2}\right)$, and the $O\left(\Delta^2/b\right)$ terms are cancelled out.

Proof of (ii). $Q(y)$ can be further rewritten as

$$Q(y) := \frac{(1+2b)\sqrt{1+3b}}{\sqrt{2\pi}}Q_0(y),$$

where

$$Q_0(y) = \frac{(1-2y)t_T + (y+b)}{\{(y+b)(1-y+b)\}^{3/2}} \exp\left[-\frac{(1+3b)\{t_T - y + (2t_T - 1)b\}^2}{2b(y+b)(1-y+b)}\right].$$

It follows that

$$Q_0^{(1)}(y) := \frac{\psi(y)}{2b\{(y+b)(1-y+b)\}^{7/2}} \exp\left[-\frac{(1+3b)\{t_T - y + (2t_T - 1)b\}^2}{2b(y+b)(1-y+b)}\right],$$

where

$$\begin{aligned}
\psi(y) &= \psi_0(y) + \psi_1(y) b + O\left(b^2\right), \\
\psi_0(y) &= -(y - t_T)\{(2t_T - 1)y - t_T\}^2, \\
\psi_1(y) &= 7(2y-1)^2 t_T^3 + \left(-20y^3 - 12y^2 + 9y + 1\right) t_T^2 \\
&\quad + \left(8y^4 + 4y^3 + 16y^2 - 5y\right) t_T - 4y^4 - 4y^2,
\end{aligned}$$

and the $O(b^2)$ rate in $\psi(y)$ is uniform on $y \in I_T$.

Heuristically, $\psi(y) \approx \psi_0(y) = -(y - t_T)\{(2t_T - 1)y - t_T\}^2$ for $b \approx 0$, and it is straightforward to see that $Q(y)$ is maximized at $y \approx t_T$. Let t_T^* solve $\psi(y) = 0$. Our argument so far suggests that $t_T^* \approx t_T$ holds for a sufficiently small $b > 0$. Then, it is reasonably conjectured that t_T^* can be expanded up to the $O(b^2)$ term, taking the form of $t_T^* = t_T + cb + O(b^2)$ for some $|c| < \infty$. The remaining task is to determine c. Substituting t_T^* into

$\psi(y)$ gives

$$\psi(t_T^*) = \psi\left\{t_T + cb + O\left(b^2\right)\right\} = 4t_T^2 (t_T - 1)^2 \{-c + 2(2t_T - 1)\} b + O\left(b^2\right).$$

It suffices to find c that makes the right-hand side at most $O(b^2)$, and this occurs when $c = 2(2t_T - 1)$. Then, we have $t_T^* = t_T + 2(2t_T - 1)b + O(b^2)$, which completes the proof. ■

S2.6. Proof of Theorem 1

It follows from Assumption 3(ii) and the definitions of $\hat{t}_B$ and t_T that $\left|\hat{t}_B - t_0\right| = \left|T^{-1}\left(\hat{t}_T\right) - T^{-1}(t_T)\right| \leq \left|\hat{t}_T - t_T\right| / \underline{T}^{(1)}$. Because $\left|\hat{t}_T - t_T\right| \leq \left|\hat{t}_T - t_T^*\right| + \left|t_T^* - t_T\right|$ and Proposition S2(ii) implies that $\left|t_T^* - t_T\right| = O(b)$, we only need to demonstrate that $\left|\hat{t}_T - t_T^*\right| = O\left(b^{1/2+\delta_1}\right)$ *a.s.* However, this statement can be established by a similar argument to the proof of Theorem 1 in Funke and Hirukawa (2025), and thus details are omitted. ■

S2.7. Proof of Theorem 2

The proof requires the following lemmata.

Lemma S4. *Put* $y = t_T$ *in* $C^{\pm}_{b,y,m}(0)$, $C^{\pm}_{b,y,m}(1)$ *and* $C^{\pm}_{b,y,m}(3)$ *defined in Lemma S2. Then, as* $n \to \infty$,

$$\begin{aligned} C^{\pm}_{b,t_T,m}(0) &= \frac{1}{2} \mp \frac{1}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + \frac{(2t_T-1)-m(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}}b^{1/2} \\ &\mp \left[\frac{m}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} + \frac{1-m^2(1-t_T)^2}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}}\right]\Delta b^{1/2} \\ &+ O\left\{\max\left(b^{3/2}, \frac{\Delta^2}{b}\right)\right\}, \qquad \text{(S24)} \\ C^{\pm}_{b,t_T,m}(1) &= -\frac{1}{\sqrt{2\pi}} \mp \frac{(2t_T-1)-m(1-t_T)}{\sqrt{2\pi}t_T(1-t_T)}\Delta + O(b), \qquad \text{(S25)} \end{aligned}$$

and

$$C^{\pm}_{b,t_T,m}(3) = -\frac{2}{\sqrt{2\pi}} + O(b). \qquad \text{(S26)}$$

Lemma S5. *As* $n \to \infty$,

$$E\left\{\frac{b^{1/2}}{\Delta}\hat{J}^{(1)}(t_T)\right\} \to \frac{d_T(2-t_T)}{\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}}.$$

Lemma S6. *As* $n \to \infty$,

$$\mathrm{Var}\left\{\sqrt{\frac{nb^{5/2}}{\Delta^2}}\hat{J}^{(1)}(t_T)\right\} \to \frac{3}{2\sqrt{\pi}\{t_T(1-t_T)\}^{5/2}}\left\{\frac{f_Y(t_T^-)+f_Y(t_T^+)}{2}\right\}.$$

Lemma S7. *If* $\left|\hat{t}_T - t_T\right| = o_p\left(b^{1/2}\right)$*, then, as* $n \to \infty$*,*

$$\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}(\varsigma) \xrightarrow{p} -\sqrt{\frac{2}{\pi}}\frac{d_T}{\left\{t_T\left(1-t_T\right)\right\}^{3/2}}$$

for any ς *on the line segment joining* $\hat{t}_T$ *and* t_T*.*

Lemma S8. *As* $n \to \infty$*,* $E|H_i|^3 = O(\Delta^3/b^4)$*.*

S2.7.1. Proof of Lemma S4

Proof of (S24). In the proof of Proposition S2(i), it has been already demonstrated that $\Phi\left(B^{\pm}_{b,y,m}\right) \to 0$ at an exponential rate. Therefore, to approximate $C^{\pm}_{b,t_T,m}(0) = \Phi\left(A^{\pm}_{b,t_T,m}\right) - \Phi\left(B^{\pm}_{b,t_T,m}\right)$, we may safely concentrate on $\Phi\left(A^{\pm}_{b,t_T,m}\right)$, where

$$A^{\pm}_{b,t_T,m} = \frac{\left[\mp\Delta + \left\{(m+2)\,t_T - (m+1)\right\}b\right]\sqrt{1+(m+3)\,b}}{\sqrt{b\left\{(t_T \pm \Delta) + (m+1)\,b\right\}\left\{1-(t_T \pm \Delta) + b\right\}}}.$$

By a second-order Taylor expansion of $\Phi\left(A^{\pm}_{b,t_T,m}\right)$ around $\Delta = 0$,

$$\Phi\left(A^{\pm}_{b,t_T,m}\right) = \Phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} + \phi\left(A^{\pm}_{b,t_T,m}\right)\frac{\partial A^{\pm}_{b,t_T,m}}{\partial \Delta}\Bigg|_{\Delta=0}\Delta + O\left(\frac{\Delta^2}{b}\right).$$

Substituting

$$A^{\pm}_{b,t_T,m}\Big|_{\Delta=0} = b^{1/2}\left\{\frac{(m+2)\,t_T - (m+1)}{\sqrt{t_T\left(1-t_T\right)}}\right\}\left\{1+O(b)\right\} \tag{S27}$$

into $\Phi\left(A^{\pm}_{b,t_T,m}\right)\big|_{\Delta=\ 0}$ and $\phi\left(A^{\pm}_{b,t_T,m}\right)\big|_{\Delta=0}$ and then using $\Phi(0)=1/2$, $\phi(0)=1/\sqrt{2\pi}$, $\phi^{(1)}(0)=\phi^{(3)}(0)=0$, $\phi^{(2)}(0)=-1/\sqrt{2\pi}$, and uniform boundedness of $\left|\phi^{(2r)}(\cdot)\right|$ for $r=1,2$, we have

$$
\begin{aligned}
&\Phi\left(A^{\pm}_{b,t_T,m}\right)\big|_{\Delta=\ 0}\\
&=\Phi\left\{b^{1/2}\left(\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right)(1+O(b))\right\}\\
&=\Phi(0)+\phi(0)\,b^{1/2}\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}\{1+O(b)\}\\
&+\frac{1}{2}\phi^{(1)}(0)\,b\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}^2\{1+O(b)\}^2+O\left(b^{3/2}\right)\\
&=\frac{1}{2}+\frac{(m+2)\,t_T-(m+1)}{\sqrt{2\pi}\sqrt{t_T\,(1-t_T)}}b^{1/2}+O\left(b^{3/2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\phi\left(A^{\pm}_{b,t_T,m}\right)\big|_{\Delta=0}\\
&=\phi\left\{b^{1/2}\left(\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right)(1+O(b))\right\}\\
&=\phi(0)+\phi^{(1)}(0)\,b^{1/2}\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}\{1+O(b)\}\\
&+\frac{1}{2}\phi^{(2)}(0)\,b\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{t_T\,(1-t_T)}}\right\}^2\{1+O(b)\}^2+O\left(b^2\right)\\
&=\frac{1}{\sqrt{2\pi}}\left[1-\frac{\{(m+2)\,t_T-(m+1)\}^2}{2t_T\,(1-t_T)}b+O\left(b^2\right)\right].
\end{aligned}
\tag{S28}
$$

It can be also shown that

$$\left.\frac{\partial A_{b,t_T,m}^{\pm}}{\partial \Delta}\right|_{\Delta=0} = \mp \frac{b^{-1/2}}{\sqrt{t_T(1-t_T)}} \left\{ 1 + \frac{2t_T^2 - 2t_T + 1 + m(1-t_T)^2}{t_T(1-t_T)} b + O\left(b^2\right) \right\}. \quad \text{(S29)}$$

A straightforward but tedious calculation finally delivers

$$\begin{aligned} \Phi\left(A_{b,t_T,m}^{\pm}\right) &= \frac{1}{2} \mp \frac{1}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} \left(\frac{\Delta}{b^{1/2}}\right) + \frac{(2t_T-1) - m(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \\ &\mp \left[\frac{m}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} + \frac{1 - m^2(1-t_T)^2}{2\sqrt{2\pi}\{t_T(1-t_T)\}^{3/2}} \right] \Delta b^{1/2} \\ &+ O\left\{ \max\left(b^{3/2}, \frac{\Delta^2}{b}\right) \right\}, \end{aligned}$$

which establishes (S24).

Proof of (S25). It follows from the identity $\phi^{(1)}(v) \equiv -v\phi(v)$ that $C_{b,t_T,m}^{\pm}(1) = \phi\left(B_{b,t_T,m}^{\pm}\right) - \phi\left(A_{b,t_T,m}^{\pm}\right)$. Since $\left|B_{b,t_T,m}^{\pm}\right| \leq O\left(b^{-1/2}\right)$ is shown in the proof of Proposition S2(i), it is the case that $\phi\left(B_{b,t_T,m}^{\pm}\right) = \phi\left(\left|B_{b,t_T,m}^{\pm}\right|\right) \to 0$ at an exponential rate. Therefore, again we may focus only on $\phi\left(A_{b,t_T,m}^{\pm}\right)$. By a second-order Taylor expansion of $\phi\left(A_{b,t_T,m}^{\pm}\right)$ around $\Delta = 0$,

$$\phi\left(A_{b,t_T,m}^{\pm}\right) = \left.\phi\left(A_{b,t_T,m}^{\pm}\right)\right|_{\Delta=0} + \phi^{(1)}\left(A_{b,t_T,m}^{\pm}\right) \left.\frac{\partial A_{b,t_T,m}^{\pm}}{\partial \Delta}\right|_{\Delta=0} \Delta + O\left(\Delta^2\right).$$

It follows from (S28) that $\phi\left(A_{b,t_T,m}^{\pm}\right)\big|_{\Delta=\ 0} = \left(1/\sqrt{2\pi}\right)\{1+O(b)\}$. Then, by $\phi^{(1)}\left(A_{b,t_T,m}^{\pm}\right) = -A_{b,t_T,m}^{\pm}\phi\left(A_{b,t_T,m}^{\pm}\right)$ and (S27),

$$\phi^{(1)}\left(A_{b,t_T,m}^{\pm}\right)\big|_{\Delta=\ 0} = -b^{1/2}\left\{\frac{(m+2)\,t_T-(m+1)}{\sqrt{2\pi}\sqrt{t_T\,(1-t_T)}}\right\}\{1+O(b)\}. \qquad \text{(S30)}$$

Therefore, by (S28)-(S30) and $\Delta^2 = o(b)$,

$$\phi\left(A_{b,t_T,m}^{\pm}\right) = \frac{1}{\sqrt{2\pi}} \pm \frac{(m+2)\,t_T-(m+1)}{\sqrt{2\pi}t_T\,(1-t_T)}\Delta + O(b)\,, \qquad \text{(S31)}$$

and thus (S25) immediately follows.

Proof of (S26). By integration by parts,

$$\begin{aligned} C_{b,t_T,m}^{\pm}(3) &= \left(B_{b,t_T,m}^{\pm}\right)^2\phi\left(B_{b,t_T,m}^{\pm}\right) \\ &\quad - \left(A_{b,t_T,m}^{\pm}\right)^2\phi\left(A_{b,t_T,m}^{\pm}\right) + 2\left\{\phi\left(B_{b,t_T,m}^{\pm}\right) - \phi\left(A_{b,t_T,m}^{\pm}\right)\right\}, \end{aligned}$$

where each of $\left(B_{b,t_T,m}^{\pm}\right)^2\phi\left(B_{b,t_T,m}^{\pm}\right)$ and $\phi\left(B_{b,t_T,m}^{\pm}\right)$ converges to zero at an exponential rate, and an approximation to $\phi\left(A_{b,t_T,m}^{\pm}\right)$ has been already derived as (S31). The remaining task is to demonstrate that

$$\left(A_{b,t_T,m}^{\pm}\right)^2\phi\left(A_{b,t_T,m}^{\pm}\right) = \mp\frac{2\{(m+2)\,t_T-(m+1)\}}{\sqrt{2\pi}t_T\,(1-t_T)}\Delta + O(b)\,, \qquad \text{(S32)}$$

because if this is true, then (S26) is immediately shown. By a second-order Taylor expansion of $\left(A^{\pm}_{b,t_T,m}\right)^2 \phi\left(A^{\pm}_{b,t_T,m}\right)$ around $\Delta = 0$,

$$\begin{aligned}
&\left(A^{\pm}_{b,t_T,m}\right)^2 \phi\left(A^{\pm}_{b,t_T,m}\right) \\
&= \left(A^{\pm}_{b,t_T,m}\right)^2 \phi\left(A^{\pm}_{b,t_T,m}\right)\Big|_{\Delta=0} \\
&+ \left\{2A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right) + \left(A^{\pm}_{b,t_T,m}\right)^2 \phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\right\} \frac{\partial A^{\pm}_{b,t_T,m}}{\partial \Delta}\Bigg|_{\Delta=0} \Delta \\
&+ O\left(\Delta^2\right).
\end{aligned} \tag{S33}$$

By (S27)-(S30), we have $A^{\pm}_{b,t_T,m}\big|_{\Delta=0} = O\left(b^{1/2}\right)$, $\phi\left(A^{\pm}_{b,t_T,m}\right)\big|_{\Delta=0} = O(1)$, $\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)\big|_{\Delta=0} = O\left(b^{1/2}\right)$, and $\partial A^{\pm}_{b,t_T,m}/\partial\Delta\big|_{\Delta=0} = O\left(b^{-1/2}\right)$. It follows from $\Delta^2 = o(b)$ that (S33) can be simplified further as

$$\left(A^{\pm}_{b,t_T,m}\right)^2 \phi\left(A^{\pm}_{b,t_T,m}\right) = 2A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right) \frac{\partial A^{\pm}_{b,t_T,m}}{\partial \Delta}\Bigg|_{\Delta=0} \Delta + O(b).$$

Then, by $A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right) = -\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right)$, (S29) and (S30),

$$\begin{aligned}
A^{\pm}_{b,t_T,m}\phi\left(A^{\pm}_{b,t_T,m}\right) \frac{\partial A^{\pm}_{b,t_T,m}}{\partial \Delta}\Bigg|_{\Delta=0} &= -\phi^{(1)}\left(A^{\pm}_{b,t_T,m}\right) \frac{\partial A^{\pm}_{b,t_T,m}}{\partial \Delta}\Bigg|_{\Delta=0} \\
&= \mp\left\{\frac{(m+2)\,t_T - (m+1)}{\sqrt{2\pi}t_T\left(1-t_T\right)}\right\}\left\{1 + O(b)\right\},
\end{aligned}$$

which implies (S32). This completes the proof. ■

S2.7.2. Proof of Lemma S5

By definition, $E\left\{\hat{J}^{(1)}(t_T)\right\} = E\left\{\dot{K}^{-}_{t_T}(Y_i)\right\} - E\left\{\dot{K}^{+}_{t_T}(Y_i)\right\}$, where, by equation (1) of the main body,

$$\begin{aligned} E\left\{\dot{K}^{\pm}_{t_T}(Y_i)\right\} &= \int_0^1 \dot{K}^{\pm}_{t_T}(u)\, f_Y(y)\, dy \\ &= \int_0^1 \dot{K}^{\pm}_{t_T}(u)\, g_Y(u)\, du + d_T \int_0^{t_T} \dot{K}^{\pm}_{t_T}(u)\, du \\ &=: E^{\pm}_{g_Y} + E^{\pm}_{d_T}. \end{aligned}$$

For $\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)$ defined in Lemma S2, $\dot{K}^{\pm}_{t_T}(u)$ can be expressed as

$$\begin{aligned} \dot{K}^{\pm}_{t_T}(u) &:= \frac{1}{b}\mathcal{L}^{\pm}_{B}(u)\, K^{\pm}_{t_T}(u) \\ &:= \frac{1}{b}\left\{\ln\left(\frac{u}{1-u}\right) - \Psi\left(p^{\pm}_{b,t_T,0}\right) + \Psi\left(q^{\pm}_{b,t_T}\right)\right\} K^{\pm}_{t_T}(u). \end{aligned}$$

Before proceeding, it is worth remarking how $\mathcal{L}^{\pm}_{B}(u)$ relates to $L_{B(x,b)}(u)$ defined in Funke and Hirukawa (2024). Let

$$\mathcal{L}^{0}_{B}(u) := \mathcal{L}^{\pm}_{B}(u)\big|_{\Delta=0} = \ln\left(\frac{u}{1-u}\right) - \Psi\left(p^{0}_{b,t_T,0}\right) + \Psi\left(q^{0}_{b,t_T}\right). \tag{S34}$$

Then, it holds that $\mathcal{L}^{0}_{B}(u) = bL_{B(t_T,b)}(u)$.

Now $E\left\{\hat{J}^{(1)}(t_T)\right\}$ can be decomposed into

$$E\left\{\hat{J}^{(1)}(t_T)\right\} = \left(E^{-}_{g_Y} - E^{+}_{g_Y}\right) + \left(E^{-}_{d_T} - E^{+}_{d_T}\right).$$

A minor modification of the proof for Theorem 2.1(i)(b) of Funke and Hirukawa (2024) gives $E_{g_Y}^{\pm} = g_Y^{(1)}(t_T) \pm g_Y^{(2)}(t_T)\,\Delta + o(\Delta)$. As a consequence, $\left|E_{g_Y}^{-} - E_{g_Y}^{+}\right|$ is at most $O(\Delta) = o\left(\Delta/b^{1/2}\right)$. In what follows, it suffices to demonstrate that

$$E_{d_T}^{-} - E_{d_T}^{+} = \frac{d_T(2-t_T)}{\sqrt{2\pi}\left\{t_T(1-t_T)\right\}^{3/2}}\left(\frac{\Delta}{b^{1/2}}\right) + o\left(\frac{\Delta}{b^{1/2}}\right). \tag{S35}$$

After substituting

$$\begin{aligned}
\ln\left(\frac{u}{1-u}\right) &= \ln\left(\frac{t_T}{1-t_T}\right) + \left\{\frac{1}{t_T(1-t_T)}\right\}(u-t_T) \\
&\quad + \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}(u-t_T)^2 + o\left(|u-t_T|^2\right), \\
\Psi\left(p_{b,t_T,0}^{\pm}\right) &= \ln\left(\frac{t_T \pm \Delta}{b}\right) + \frac{b}{2(t_T \pm \Delta)} + O\left(b^2\right),
\end{aligned}$$

and

$$\Psi\left(q_{b,t_T}^{\pm}\right) = \ln\left\{\frac{1-(t_T \pm \Delta)}{b}\right\} + \frac{b}{2\left\{1-(t_T \pm \Delta)\right\}} + O\left(b^2\right)$$

by (S5) into $\mathcal{L}_B^{\pm}(u)$, using

$$\begin{aligned}
\ln(t_T \pm \Delta) - \ln t_T &= \pm\frac{\Delta}{t_T} - \frac{\Delta^2}{2t_T^2} \pm \frac{\Delta^3}{3t_T^3} + O\left(\Delta^4\right), \\
\ln\left\{1-(t_T \pm \Delta)\right\} - \ln(1-t_T) &= \mp\frac{\Delta}{1-t_T} - \frac{\Delta^2}{2(1-t_T)^2} \\
&\quad \mp \frac{\Delta^3}{3(1-t_T)^3} + O\left(\Delta^4\right),
\end{aligned}$$

and

$$\frac{1}{1-(t_T \pm \Delta)} - \frac{1}{t_T \pm \Delta} = \frac{2t_T - 1}{t_T(1-t_T)} \pm \frac{2t_T^2 - 2t_T + 1}{t_T^2 (1-t_T)^2}\Delta + O\left(\Delta^2\right),$$

and recognizing that $O(\Delta^4)$ and $O(\Delta^2 b)$ terms are at most $o(b^2)$, $\mathcal{L}_B^{\pm}(u)$ can be rearranged as

$$\mathcal{L}_B^{\pm}(u) := L^{\pm} + \left\{\frac{1}{t_T(1-t_T)}\right\}(u-t_T) + \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}(u-t_T)^2,$$

where

$$\begin{aligned} L^{\pm} = &\mp \frac{\Delta}{t_T(1-t_T)} - \frac{2t_T-1}{2t_T^2(1-t_T)^2}\Delta^2 \mp \frac{3t_T^2 - 3t_T + 1}{3t_T^3(1-t_T)^3}\Delta^3 \\ &+ \frac{2t_T-1}{2t_T(1-t_T)} b \pm \frac{2t_T^2 - 2t_T + 1}{2t_T^2(1-t_T)^2}\Delta b + O\left(b^2\right). \end{aligned} \tag{S36}$$

It follows that

$$\begin{aligned} E_{d_T}^{\pm} = \frac{d_T}{b}\Bigg[&L^{\pm}\int_0^{t_T} K_{t_T}^{\pm}(u)\,du + \left\{\frac{1}{t_T(1-t_T)}\right\}\int_0^{t_T}(u-t_T)K_{t_T}^{\pm}(u)\,du \\ &+ \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}\int_0^{t_T}(u-t_T)^2 K_{t_T}^{\pm}(u)\,du\Bigg]. \end{aligned} \tag{S37}$$

Now

$$\int_0^{t_T} u^m K_{t_T}^{\pm}(u)\,du = \frac{B\left(p_{b,t_T,m}^{\pm}, q_{b,t_T}^{\pm}\right)}{B\left(p_{b,t_T,0}^{\pm}, q_{b,t_T}^{\pm}\right)}\int_0^{t_T} f_{Z_{b,t_T,m}^{\pm}}(z)\,dz,$$

where $f_{Z^{\pm}_{b,t_T,m}}(z)$ is the pdf of the beta random variable $Z^{\pm}_{b,t_T,m}$, and

$$\frac{B\left(p^{\pm}_{b,t_T,m}, q^{\pm}_{b,t_T}\right)}{B\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)} = \begin{cases} 1 & \text{for } m = 0 \\ \dfrac{\prod_{k=1}^{m}\left(\frac{t_T \pm \Delta}{b} + k\right)}{\prod_{k=1}^{m}\left(\frac{1}{b} + k + 1\right)} & \text{for } m \geq 1 \end{cases}.$$

by the property of the beta function. Applying Lemmata S2 and S4 and then making straightforward but tedious calculations, we also have, as $n \to \infty$,

$$\begin{aligned} &\int_0^{t_T} f_{Z^{\pm}_{b,t_T,m}}(z)\, dz \\ &:= \Lambda^{\pm}_b - \frac{m(1-t_T)}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \mp \frac{m(2-m)(1-t_T)^2}{2\sqrt{2\pi}\left\{t_T(1-t_T)\right\}^{3/2}} \Delta b^{1/2}, \end{aligned} \tag{S38}$$

where

$$\begin{aligned} \Lambda^{\pm}_b = &\ \frac{1}{2} \mp \frac{1}{\sqrt{2\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + \frac{2(2t_T-1)}{3\sqrt{2\pi}\sqrt{t_T(1-t_T)}} b^{1/2} \\ &\mp \frac{2+3(2t_T-1)^2}{3\sqrt{2\pi}\left(t_T(1-t_T)\right)^{3/2}} \Delta b^{1/2} + O\left(\frac{\Delta^2}{b}\right) + O(b) + O\left(b^{3/2}\right). \end{aligned} \tag{S39}$$

The above results provide approximations to $\int_0^{t_T} u^m K^{\pm}_{t_T}(u)\, du$ for $m = 0, 1, 2$. Obviously, for $m = 0$,

$$\int_0^{t_T} K^{\pm}_{t_T}(u)\, du = \int_0^{t_T} f_{Z^{\pm}_{b,t_T,0}}(z)\, dz = \Lambda^{\pm}_b. \tag{S40}$$

For $m = 1$,

$$\begin{aligned}
&\int_0^{t_T} u K_{t_T}^{\pm}(u)\, du \\
&= \frac{(t_T \pm \Delta)/b + 1}{1/b + 2} \int_0^{t_T} f_{Z_{b,t_T,1}^{\pm}}(z)\, dz \\
&= \left\{ t_T \pm \Delta - (2t_T - 1) b \mp 2\Delta b + O\left(b^2\right) \right\} \\
&\times \left[\Lambda_b^{\pm} - \frac{1 - t_T}{\sqrt{2\pi}\sqrt{t_T(1 - t_T)}} b^{1/2} \mp \frac{(1 - t_T)^2}{2\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{3/2}} \Delta b^{1/2} \right].
\end{aligned}$$

Finally, for $m = 2$,

$$\begin{aligned}
&\int_0^{t_T} u^2 K_{t_T}^{\pm}(u)\, du \\
&= \frac{\left\{(t_T \pm \Delta)/b + 1\right\}\left\{(t_T \pm \Delta)/b + 2\right\}}{(1/b + 2)(1/b + 3)} \int_0^{t_T} f_{Z_{b,t_T,2}^{\pm}}(z)\, dz \\
&= \left\{ t_T^2 \pm 2t_T\Delta + \left(3t_T - 5t_T^2\right) b + \Delta^2 \pm (3 - 10t_T)\Delta b + O\left(b^2\right) \right\} \\
&\times \left\{ \Lambda_b^{\pm} - \frac{2(1 - t_T)}{\sqrt{2\pi}\sqrt{t_T(1 - t_T)}} b^{1/2} \right\}.
\end{aligned}$$

After straightforward but tedious calculations, we obtain

$$\begin{aligned}
&\int_0^{t_T} (u - t_T) K_{t_T}^{\pm}(u)\, du \\
&= \left\{ \pm\Delta - (2t_T - 1) b \mp 2\Delta b + O\left(b^2\right) \right\} \Lambda_b^{\pm} \\
&- \frac{t_T(1 - t_T)}{\sqrt{2\pi}\sqrt{t_T(1 - t_T)}} b^{1/2} \mp \frac{2t_T(1 - t_T)^2}{3\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{3/2}} \Delta b^{1/2} + O\left(b^{3/2}\right) \qquad \text{(S41)}
\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^{t_T} (u - t_T)^2 K_{t_T}^{\pm}(u)\, du \\
&= \left\{ t_T(1 - t_T)\, b + \Delta^2 \pm 3(1 - 2t_T)\Delta b + O\left(b^2\right) \right\} \Lambda_b^{\pm} \\
&\mp \frac{t_T(1 - t_T)}{\sqrt{2\pi}\sqrt{t_T(1 - t_T)}} \Delta b^{1/2} + O\left(b^{3/2}\right). \qquad \text{(S42)}
\end{aligned}
$$

Substituting (S36), (S40), (S41), and (S42) into (S37) and simplifying this yield

$$
\begin{aligned}
E_{d_T}^{\pm} = \frac{d_T}{b} \Bigg\{ & \underbrace{O\left(\Delta^2\right)}_{=o\left(\Delta b^{1/2}\right)} \mp \underbrace{\frac{1 - 2t_T}{2\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{5/2}} \left(\frac{\Delta^3}{b^{1/2}}\right)}_{=o\left(\Delta b^{1/2}\right)} + \underbrace{O(\Delta b)}_{=o\left(\Delta b^{1/2}\right)} + \underbrace{O\left(\Delta^3\right)}_{=o\left(\Delta b^{1/2}\right)} \\
& + \underbrace{O\left(b^2\right)}_{=o\left(\Delta b^{1/2}\right)} + O\left(b^{1/2}\right) \mp \frac{2 - t_T}{2\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{3/2}} \Delta b^{1/2} + \underbrace{O\left(b^{3/2}\right)}_{=o\left(\Delta b^{1/2}\right)} \Bigg\} \\
= \frac{d_T}{b} \Bigg\{ & \mp \frac{2 - t_T}{2\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{3/2}} \Delta b^{1/2} + o\left(\Delta b^{1/2}\right) + O\left(b^{1/2}\right) \Bigg\}.
\end{aligned}
$$

Finally, (S35) can be established by recognizing that $O\left(b^{1/2}\right)$ terms inside the brackets of $E_{d_T}^{\pm}$ are cancelled out after taking the difference $E_{d_T}^{-} - E_{d_T}^{+}$. This completes the proof. ■

S2.7.3. Proof of Lemma S6

Because

$$Var\left\{\hat{J}^{(1)}\left(t_T\right)\right\} = \frac{1}{n}Var\left(H_i\right) = \frac{1}{n}\left\{E\left(H_i^2\right) - E^2\left(H_i\right)\right\},$$

and $|E\left(H_i\right)| = \left|E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}\right| = O\left(\Delta/b^{1/2}\right)$ as shown above, we concentrate on approximating

$$E\left(H_i^2\right) = \int_0^1 \left\{\dot{K}_{t_T}^{-}\left(u\right) - \dot{K}_{t_T}^{+}\left(u\right)\right\}^2 f_Y\left(u\right) du.$$

However, it is not obvious whether the right-hand side may be safely rewritten as

$$\left\{\frac{f_Y(t_T^-) + f_Y(t_T^+)}{2}\right\} \int_0^1 \left\{\dot{K}_{t_T}^{-}\left(u\right) - \dot{K}_{t_T}^{+}\left(u\right)\right\}^2 du,$$

like the cases in which standard symmetric kernels are employed. It will shortly turn out that this is also valid in our case. Using equation (1) of the main body, we decompose $E\left(H_i^2\right)$ into two parts:

$$\begin{aligned} E\left(H_i^2\right) &= \int_0^1 \left\{\dot{K}_{t_T}^{-}\left(u\right) - \dot{K}_{t_T}^{+}\left(u\right)\right\}^2 g_Y\left(u\right) du \\ &\quad + d_T \int_0^{t_T} \left\{\dot{K}_{t_T}^{-}\left(u\right) - \dot{K}_{t_T}^{+}\left(u\right)\right\}^2 du \\ &=: V_{g_Y} + V_{d_T}. \end{aligned}$$

Then, the proof boils down to demonstrating the following two statements:

$$V_{g_Y} \sim g_Y(t_T)\,\frac{3}{2\sqrt{\pi}\,\{t_T(1-t_T)\}^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right). \tag{S43}$$

$$V_{d_T} \sim \left(\frac{d_T}{2}\right)\frac{3}{2\sqrt{\pi}\,\{t_T(1-t_T)\}^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right). \tag{S44}$$

It should be recognized that (S43) and (S44) jointly establish the lemma, because $g_Y(t_T) + d_T/2 = \left\{f_Y(t_T^-) + f_Y(t_T^+)\right\}/2$.

Proof of (S43). V_{g_Y} can be further decomposed into $V_{g_Y} := V_{g_Y}^{2-} + V_{g_Y}^{2+} - 2V_{g_Y}^{+-}$, where

$$V_{g_Y}^{2\pm} = \int_0^1 \left\{\dot{K}_{t_T}^{\pm}(u)\right\}^2 g_Y(u)\,du$$

and

$$V_{g_Y}^{+-} = \int_0^1 \dot{K}_{t_T}^{-}(u)\,\dot{K}_{t_T}^{+}(u)\,g_Y(u)\,du.$$

In what follows, approximations to $V_{g_Y}^{2\pm}$ and $V_{g_Y}^{+-}$ are derived separately, and (S43) is finally obtained.

(i) Approximation to $V_{g_Y}^{2\pm}$. Observe that

$$\begin{aligned} V_{g_Y}^{2\pm} &= \frac{1}{b^2}\int_0^1 \left\{\mathcal{L}_B^{\pm}(u)\,K_{t_T}^{\pm}(u)\right\}^2 g_Y(u)\,du \\ &= \frac{1}{b^2}\int_0^1 \left\{\ln\left(\frac{u}{1-u}\right) - \Psi\left(p_{b,t_T,0}^{\pm}\right) + \Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 \\ &\quad \times \left\{K_{t_T}^{\pm}(u)\right\}^2 g_Y(u)\,du, \end{aligned} \tag{S45}$$

where the integral part can be alternatively expressed as

$$G_b^{\pm}(t_T)\, E\left[\left\{\ln\left(\frac{Z_{b/2,t_T,0}^{\pm}}{1-Z_{b/2,t_T,0}^{\pm}}\right)-\Psi\left(p_{b,t_T,0}^{\pm}\right)+\Psi\left(q_{b,t_T}^{\pm}\right)\right\}^2 g_Y\left(Z_{b/2,t_T,0}^{\pm}\right)\right]$$

for the beta random variable $Z_{b/2,t_T,m}^{\pm}$ and

$$G_b^{\pm}(t_T) := \frac{B\left(p_{b/2,t_T,0}^{\pm}, q_{b/2,t_T}^{\pm}\right)}{B^2\left(p_{b,t_T,0}^{\pm}, q_{b,t_T}^{\pm}\right)}.$$

Then, by the property of the beta function, definitions of $\left(p_{b,t_T,0}^{\pm}, q_{b,t_T}^{\pm}\right)$, (S1) and $b = o(\Delta)$,

$$\begin{aligned} G_b^{\pm}(t_T) &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{(t_T \pm \Delta)\{1-(t_T \pm \Delta)\}}}\{1+O(b)\} \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T(1-t_T)}}\{1+O(\Delta)\}. \end{aligned} \tag{S46}$$

Furthermore, a mean-value expansion of $g_Y\left(Z_{b/2,t_T,0}^{\pm}\right)$ around $Z_{b/2,t_T,0}^{\pm} =$

t_T leads to

$$
\begin{aligned}
&E\left[\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2 g_Y\left(Z^{\pm}_{b/2,t_T,0}\right)\right]\\
&=g_Y\left(t_T\right)E\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2\\
&+E\left[\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2 g_Y^{(1)}\left(\bar{t}^{\pm}_T\right)\left(Z^{\pm}_{b/2,t_T,0}-t_T\right)\right]\\
&=: W_1^{\pm}+W_2^{\pm}
\end{aligned}
$$

for some $\bar{t}^{\pm}_T$ on the line segment joining $Z^{\pm}_{b/2,t_T,0}$ and t_T. We work on $W_1^{\pm}$ first. It follows from definitions of $\left(p^{\pm}_{b,t_T,0}, q^{\pm}_{b,t_T}\right)$ and (S3) that

$$
\begin{aligned}
W_1^{\pm}=g_Y\left(t_T\right)&\left[\left\{\Psi\left(\frac{2\left(t_T\pm\Delta\right)}{b}+1\right)-\Psi\left(\frac{2\left(1-\left(t_T\pm\Delta\right)\right)}{b}+1\right)\right.\right.\\
&\left.-\Psi\left(\frac{t_T\pm\Delta}{b}+1\right)+\Psi\left(\frac{1-\left(t_T\pm\Delta\right)}{b}+1\right)\right\}^2\\
&\left.+\Psi^{(1)}\left\{\frac{2\left(t_T\pm\Delta\right)}{b}+1\right\}+\Psi^{(1)}\left\{\frac{2\left(1-\left(t_T\pm\Delta\right)\right)}{b}+1\right\}\right].
\end{aligned}
$$

Then, by (S5) and (S6),

$$
\begin{aligned}
&\left[\Psi\left\{\frac{2\left(t_T\pm\Delta\right)}{b}+1\right\}-\Psi\left\{\frac{2\left(1-\left(t_T\pm\Delta\right)\right)}{b}+1\right\}\right.\\
&\left.-\Psi\left(\frac{t_T\pm\Delta}{b}+1\right)+\Psi\left\{\frac{1-\left(t_T\pm\Delta\right)}{b}+1\right\}\right]^2=O\left(b^2\right),
\end{aligned}
$$

and

$$\Psi^{(1)}\left\{\frac{2\left(t_T\pm\Delta\right)}{b}+1\right\}+\Psi^{(1)}\left\{\frac{2\left(1-\left(t_T\pm\Delta\right)\right)}{b}+1\right\}$$
$$=\frac{b}{2\left(t_T\pm\Delta\right)\left\{1-\left(t_T\pm\Delta\right)\right\}}\left\{1+O\left(b\right)\right\}.$$

It follows from $b=o\left(\Delta\right)$ that

$$W_1^{\pm}=\frac{g_Y\left(t_T\right)}{2\left(t_T\pm\Delta\right)\left\{1-\left(t_T\pm\Delta\right)\right\}}b\left\{1+O\left(b\right)\right\}=\frac{bg_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1+O\left(\Delta\right)\right\}.$$

The remaining task is to show that $\left|W_2^{\pm}\right|$ is of smaller order in magnitude than $\left|W_1^{\pm}\right|$. To be more precise, we demonstrate that $\left|W_2^{\pm}\right|=O\left(b^{3/2}\right)$. To see this, observe that

$$\begin{aligned}
&\left|W_2^{\pm}\right| \\
&\leq E\left\{\left|\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right|^2\left|g_Y^{(1)}\left(\bar{t}_T^{\pm}\right)\right|\left|Z^{\pm}_{b/2,t_T,0}-t_T\right|\right\} \\
&\leq \sup_{y\in[0,1]}\left|g_Y^{(1)}\left(y\right)\right|\left[E\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2\right]^{1/2} \\
&\times\left[E\left\{\left(\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right)^2\left(Z^{\pm}_{b/2,t_T,0}-t_T\right)^2\right\}\right]^{1/2}
\end{aligned}$$

by the Cauchy-Schwarz inequality. Now $\sup_{y\in[0,1]}\left|g_Y^{(1)}\left(y\right)\right|<\infty$ by Assumption 2, and $E\left\{\ln\left(Z^{\pm}_{b/2,t_T,0}/\left(1-Z^{\pm}_{b/2,t_T,0}\right)\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2=O\left(b\right)$

as demonstrated above. By (S2)-(S6), we can also obtain

$$E\left[\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2\left(Z^{\pm}_{b/2,t_T,0}-t_T\right)^2\right]=O\left(b^2\right),$$

and thus $\left|W_2^{\pm}\right|=O\left(b^{3/2}\right)$ follows.

Therefore, by $\Delta=o\left(b^{1/2}\right)$,

$$\begin{aligned}&E\left[\left\{\ln\left(\frac{Z^{\pm}_{b/2,t_T,0}}{1-Z^{\pm}_{b/2,t_T,0}}\right)-\Psi\left(p^{\pm}_{b,t_T,0}\right)+\Psi\left(q^{\pm}_{b,t_T}\right)\right\}^2 g_Y\left(Z^{\pm}_{b/2,t_T,0}\right)\right]\\&=\frac{bg_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1+O\left(\Delta\right)\right\}+O\left(b^{3/2}\right)\\&=\frac{bg_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1+O\left(b^{1/2}\right)\right\}.\end{aligned}\tag{S47}$$

Substituting (S46) and (S47) into (S45) and recognizing that $\Delta=o\left(b^{1/2}\right)$, we may conclude that

$$\begin{aligned}V^{2\pm}_{g_Y}&=\frac{1}{b^2}\left[\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left\{1+O\left(\Delta\right)\right\}\right]\left[\frac{bg_Y\left(t_T\right)}{2t_T\left(1-t_T\right)}\left\{1+O\left(b^{1/2}\right)\right\}\right]\\&=\frac{b^{-3/2}}{4\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left\{g_Y\left(t_T\right)+O\left(b^{1/2}\right)\right\}.\end{aligned}$$

(ii) Approximation to $V^{+-}_{g_Y}$. Next, the integral part of

$$V^{+-}_{g_Y}=\frac{1}{b^2}\int_0^1\mathcal{L}^-_B\left(u\right)\mathcal{L}^+_B\left(u\right)K^-_{t_T}\left(u\right)K^+_{t_T}\left(u\right)g_Y\left(u\right)du\tag{S48}$$

can be rewritten as

$$G_b^0(t_T)\,E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^-_{b,t_T,0}\right)+\Psi\left(q^-_{b,t_T}\right)\right\}\right.$$
$$\left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^+_{b,t_T,0}\right)+\Psi\left(q^+_{b,t_T}\right)\right\}g_Y\left(Z^0_{b/2,t_T,0}\right)\right],$$

for the beta random variable $Z^0_{b/2,t_T,0}$ and

$$G_b^0(t_T):=\frac{B\left(p^0_{b/2,t_T,0},q^0_{b/2,t_T}\right)}{B\left(p^-_{b,t_T,0},q^-_{b,t_T}\right)B\left(p^+_{b,t_T,0},q^+_{b,t_T}\right)}.$$

Then, by the property of the beta function, definitions of $\left(p^\pm_{b,t_T,0},q^\pm_{b,t_T}\right)$ and (S1),

$$\begin{aligned}
&G_b^0(t_T)\\
&=\frac{b^{-1/2}}{2\sqrt{\pi}}\left(\frac{t_T}{t_T^2-\Delta^2}\right)^{1/2}\left\{\frac{1-t_T}{(1-t_T)^2-\Delta^2}\right\}^{1/2}\left(\frac{t_T}{t_T-\Delta}\right)^{(t_T-\Delta)/b}\left(\frac{t_T}{t_T+\Delta}\right)^{(t_T+\Delta)/b}\\
&\times\left(\frac{1-t_T}{1-t_T+\Delta}\right)^{(1-t_T+\Delta)/b}\left(\frac{1-t_T}{1-t_T-\Delta}\right)^{(1-t_T-\Delta)/b}\{1+O(b)\}\\
&=\frac{b^{-1/2}}{2\sqrt{\pi}}\left(\frac{t_T}{t_T^2-\Delta^2}\right)^{1/2}\left\{\frac{1-t_T}{(1-t_T)^2-\Delta^2}\right\}^{1/2}\frac{1}{(1-\Delta^2/t_T^2)^{t_T/b}}\left(1-\frac{2\Delta}{t_T+\Delta}\right)^{\Delta/b}\\
&\times\frac{1}{\left\{1-\Delta^2/(1-t_T)^2\right\}^{(1-t_T)/b}}\left(1-\frac{2\Delta}{1-t_T+\Delta}\right)^{\Delta/b}\{1+O(b)\}.
\end{aligned}$$

Because

$$\left(\frac{t_T}{t_T^2-\Delta^2}\right)^{1/2}\left\{\frac{1-t_T}{(1-t_T)^2-\Delta^2}\right\}^{1/2} = \frac{1+O\left(\Delta^2\right)}{\sqrt{t_T\left(1-t_T\right)}},$$

$$\frac{1}{\left(1-\Delta^2/t_T^2\right)^{t_T/b}}\left(1-\frac{2\Delta}{t_T+\Delta}\right)^{\Delta/b} = \frac{1-\left(2/t_T\right)\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)}{1-\left(1/t_T\right)\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)} = 1-\frac{1}{t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right),$$

$$\frac{1}{\left\{1-\Delta^2/\left(1-t_T\right)^2\right\}^{(1-t_T)/b}}\left(1-\frac{2\Delta}{1-t_T+\Delta}\right)^{\Delta/b} = \frac{1-\left\{2/\left(1-t_T\right)\right\}\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)}{1-\left\{1/\left(1-t_T\right)\right\}\left(\Delta^2/b\right)+o\left(\Delta^2/b\right)} = 1-\frac{1}{1-t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right),$$

and $b = o\left(\Delta^2/b\right)$, we have

$$\begin{aligned} G_b^0\left(t_T\right) &= \frac{b^{-1/2}\left\{1+O\left(\Delta^2\right)\right\}}{2\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left\{1-\frac{1}{t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\ &\quad\times\left\{1-\frac{1}{1-t_T}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}\left\{1+O\left(b\right)\right\} \\ &= \frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left\{1-\frac{1}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}. \end{aligned} \tag{S49}$$

Moreover, by a mean-value expansion of $g_Y\left(Z^0_{b/2,t_T,0}\right)$ around $Z^0_{b/2,t_T,0} =$

t_T,

$$
\begin{aligned}
&E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^-_{b,t_T,0}\right)+\Psi\left(q^-_{b,t_T}\right)\right\}\right.\\
&\left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^+_{b,t_T,0}\right)+\Psi\left(q^+_{b,t_T}\right)\right\}g_Y\left(Z^0_{b/2,t_T,0}\right)\right]\\
&=g_Y\left(t_T\right)E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^-_{b,t_T,0}\right)+\Psi\left(q^-_{b,t_T}\right)\right\}\right.\\
&\left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^+_{b,t_T,0}\right)+\Psi\left(q^+_{b,t_T}\right)\right\}\right]\\
&+E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^-_{b,t_T,0}\right)+\Psi\left(q^-_{b,t_T}\right)\right\}\right.\\
&\left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^+_{b,t_T,0}\right)+\Psi\left(q^+_{b,t_T}\right)\right\}g^{(1)}_Y\left(\bar{t}^0_T\right)\left(Z^0_{b/2,t_T,0}-t_T\right)\right]\\
&=:W^0_1+W^0_2
\end{aligned}
$$

for some $\bar{t}^0_T$ on the line segment joining $Z^0_{b/2,t_T,0}$ and t_T. For W^0_1, by (S2), (S3), (S5), and (S6),

$$
\begin{aligned}
&E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^-_{b,t_T,0}\right)+\Psi\left(q^-_{b,t_T}\right)\right\}\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right)-\Psi\left(p^+_{b,t_T,0}\right)+\Psi\left(q^+_{b,t_T}\right)\right\}\right]\\
&=\left[\Psi\left(\frac{2t_T}{b}+1\right)-\Psi\left\{\frac{2\left(1-t_T\right)}{b}+1\right\}-\Psi\left(\frac{t_T-\Delta}{b}+1\right)+\Psi\left\{\frac{1-\left(t_T-\Delta\right)}{b}+1\right\}\right]\\
&\times\left[\Psi\left(\frac{2t_T}{b}+1\right)-\Psi\left\{\frac{2\left(1-t_T\right)}{b}+1\right\}-\Psi\left(\frac{t_T+\Delta}{b}+1\right)+\Psi\left\{\frac{1-\left(t_T+\Delta\right)}{b}+1\right\}\right]\\
&+\Psi^{(1)}\left(\frac{2t_T}{b}+1\right)+\Psi^{(1)}\left\{\frac{2\left(1-t_T\right)}{b}+1\right\}\\
&=\frac{b}{2t_T\left(1-t_T\right)}\left\{1-\frac{2}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}.
\end{aligned}
$$

Therefore,

$$W_1^0 = \frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 - \frac{2}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}.$$

It can be also shown that $|W_2^0| = O\left(b^{3/2}\right)$, and thus

$$\begin{aligned}
&E\left[\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right) - \Psi\left(p^-_{b,t_T,0}\right) + \Psi\left(q^-_{b,t_T}\right)\right\}\right.\\
&\left.\times\left\{\ln\left(\frac{Z^0_{b/2,t_T,0}}{1-Z^0_{b/2,t_T,0}}\right) - \Psi\left(p^+_{b,t_T,0}\right) + \Psi\left(q^+_{b,t_T}\right)\right\} g_Y\left(Z^0_{b/2,t_T,0}\right)\right]\\
&= \frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 - \frac{2}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} + O\left(b^{3/2}\right)\\
&= \frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 - \frac{2}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}
\end{aligned} \tag{S50}$$

by recognizing that $b^{1/2} = o\left(\Delta^2/b\right)$. Substituting (S49) and (S50) into (S48) finally yields

$$\begin{aligned}
V^{+-}_{g_Y} &= \frac{1}{b^2}\left\{\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T(1-t_T)}}\right\}\left\{1 - \frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}\\
&\times \frac{bg_Y(t_T)}{2t_T(1-t_T)}\left\{1 - \frac{2}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}\\
&= \frac{b^{-3/2}}{4\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{3/2}}\left\{g_Y(t_T) - \frac{3g_Y(t_T)}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}.
\end{aligned}$$

(iii) Proof of (S43). Combining two approximations above and again using $b^{1/2} = o\left(\Delta^2/b\right)$ yield

$$\begin{aligned}
V_{g_Y} &= \left(V_{g_Y}^{2-} + V_{g_Y}^{2+}\right) - 2V_{g_Y}^{+-} \\
&= \frac{b^{-3/2}}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left\{g_Y\left(t_T\right) + O\left(b^{1/2}\right)\right\} \\
&- \frac{b^{-3/2}}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{3/2}}\left\{g_Y\left(t_T\right) - \frac{3g_Y\left(t_T\right)}{t_T\left(1-t_T\right)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \\
&= \frac{3g_Y\left(t_T\right)}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right) + o\left(\frac{\Delta^2}{b^{5/2}}\right).
\end{aligned}$$

Therefore, (S43) is established.

Proof of (S44). Let

$$V_{d_T}^{2\pm} := d_T \int_0^{t_T} \left\{\dot{K}_{t_T}^{\pm}\left(u\right)\right\}^2 du$$

and

$$V_{d_T}^{+-} := d_T \int_0^{t_T} \dot{K}_{t_T}^{-}\left(u\right)\dot{K}_{t_T}^{+}\left(u\right) du.$$

Then, it holds that $V_{d_T} = V_{d_T}^{2-} + V_{d_T}^{2+} - 2V_{d_T}^{+-}$. As above, $V_{d_T}^{2\pm}$ and $V_{d_T}^{+-}$ are approximated separately.

(i) Approximation to $V_{d_T}^{2\pm}$. Given the notation in the proof of (S43), $V_{d_T}^{2\pm}$ can be expressed as

$$
\begin{aligned}
V_{d_T}^{2\pm} &= \frac{d_T}{b^2}\int_0^{t_T}\left\{\mathcal{L}_B^{\pm}(u)\,K_{t_T}^{\pm}(u)\right\}^2 du \\
&= \frac{d_T G_b^{\pm}(t_T)}{b^2}\int_0^{t_T}\left\{\mathcal{L}_B^{\pm}(z)\right\}^2 f_{Z_{b/2,t_T,0}^{\pm}}(z)\,dz, \qquad \text{(S51)}
\end{aligned}
$$

where $f_{Z_{b/2,t_T,m}^{\pm}}(z)$ is the pdf of the beta random variable $Z_{b/2,t_T,m}^{\pm}$. Below the integral part is approximated. After substituting

$$
\begin{aligned}
&\ln\left(\frac{z}{1-z}\right)\\
&= \ln\left(\frac{t_T}{1-t_T}\right) + \left\{\frac{1}{t_T(1-t_T)}\right\}(z-t_T) + \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}(z-t_T)^2\\
&+ o\left(|z-t_T|^2\right),\\
&\ln^2\left(\frac{z}{1-z}\right)\\
&= \ln^2\left(\frac{t_T}{1-t_T}\right) + \left[\frac{2\ln\{t_T/(1-t_T)\}}{t_T(1-t_T)}\right](z-t_T)\\
&+ \left[\frac{1+(2t_T-1)\ln\{t_T/(1-t_T)\}}{t_T^2(1-t_T)^2}\right](z-t_T)^2 + o\left(|z-t_T|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Psi\left(p_{b,t_T,0}^{\pm}\right) - \Psi\left(q_{b,t_T}^{\pm}\right)\\
&= \ln\left(\frac{t_T}{1-t_T}\right) \mp \frac{\Delta}{t_T(1-t_T)} + \left\{\frac{2t_T-1}{2t_T^2(1-t_T)^2}\right\}\Delta^2\\
&- \left\{\frac{2t_T-1}{2t_T(1-t_T)}\right\} b + O(\Delta b)
\end{aligned}
$$

by (S5) into $\left\{\mathcal{L}_B^{\pm}(z)\right\}^2$ and making straightforward but tedious calculations, we can rewrite $\left\{\mathcal{L}_B^{\pm}(z)\right\}^2$ as

$$
\begin{aligned}
&\left\{\mathcal{L}_B^{\pm}(z)\right\}^2 \\
&= \left[\mp \frac{2\Delta}{t_T^2 (1-t_T)^2} + \left\{\frac{2t_T - 1}{t_T^2 (1-t_T)^2}\right\} b + O\left(\Delta^2\right)\right](z - t_T) \\
&+ \left[\frac{1}{t_T^2 (1-t_T)^2} \mp \left\{\frac{2t_T - 1}{t_T^3 (1-t_T)^3}\right\}\Delta + \left\{\frac{(2t_T - 1)^2}{2t_T^3 (1-t_T)^3}\right\} b + O\left(\Delta^2\right)\right](z - t_T)^2 \\
&+ \frac{\Delta^2}{t_T^2 (1-t_T)^2} + o\left(\Delta^2\right). \qquad \text{(S52)}
\end{aligned}
$$

It follows from (S38) and (S39) that for $m \geq 0$,

$$
\int_0^{t_T} f_{Z^{\pm}_{b/2, t_T, m}}(z)\, dz = \frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T (1-t_T)}} \left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right). \qquad \text{(S53)}
$$

Then, as in the proof of Lemma S5, approximations to $\int_0^{t_T} z^m f_{Z^{\pm}_{b/2, t_T, 0}}(z)\, dz$ for $m = 1, 2$ can be derived. For $m = 1$,

$$
\begin{aligned}
&\int_0^{t_T} z f_{Z^{\pm}_{b/2, t_T, 0}}(z)\, dz \\
&= \left\{t_T \pm \Delta + \left(\frac{1 - 2t_T}{2}\right) b \mp \Delta b + O\left(b^2\right)\right\} \\
&\times \left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T (1-t_T)}} \left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}.
\end{aligned}
$$

For $m=2$,

$$\int_0^{t_T} z^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz$$
$$= \left\{t_T^2 \pm 2t_T\Delta + \left(\frac{3t_T-5t_T^2}{2}\right)b + \Delta^2 \pm \left(\frac{3-10t_T}{2}\right)\Delta b + O\left(b^2\right)\right\}$$
$$\times \left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}.$$

Hence,

$$\int_0^{t_T} \left(z-t_T\right) f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz$$
$$= \left\{\pm\Delta + \left(\frac{1-2t_T}{2}\right)b \mp \Delta b + O\left(b^2\right)\right\}$$
$$\times \left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}, \tag{S54}$$

and

$$\int_0^{t_T} \left(z-t_T\right)^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz$$
$$= \left[\left\{\frac{t_T\left(1-t_T\right)}{2}\right\}b + \Delta^2 \pm \left\{\frac{3\left(1-2t_T\right)}{2}\right\}\Delta b + O\left(b^2\right)\right]$$
$$\times \left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}\left(\frac{\Delta}{b^{1/2}}\right) + O\left(b^{1/2}\right)\right\}. \tag{S55}$$

Combining (S52)-(S55) and using the fact that $O\left(b^{1/2}\right)$ and $O\left(\Delta\right)$ terms are at most $o(\Delta^2/b)$ offer

$$\int_0^{t_T} \left\{\mathcal{L}_B^{\pm}(z)\right\}^2 f_{Z^{\pm}_{b/2,t_T,0}}(z)\,dz$$
$$= \frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}. \tag{S56}$$

In the end,

$$V_{d_T}^{2\pm} = \frac{d_T}{b^2}\left\{\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{(t_T\pm\Delta)\{1-(t_T\pm\Delta)\}}}\right\}\{1+O(b)\}$$
$$\times \frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + o\left(\frac{\Delta^2}{b}\right)\right\}$$
$$= \frac{b^{-3/2}d_T}{4\sqrt{\pi}\{t_T(1-t_T)\}^{3/2}}$$
$$\times \left\{\frac{1}{2} \mp \frac{1}{\sqrt{\pi}\sqrt{t_T(1-t_T)}}\left(\frac{\Delta}{b^{1/2}}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \tag{S57}$$

can be obtained by substituting (S46) and (S56) into (S51) and recognizing that $O(b)$ and $O(\Delta)$ terms are at most $o(\Delta^2/b)$.

(ii) Approximation to $V_{d_T}^{+-}$. Our next task is to find an approximation to

$$V_{d_T}^{+-} = \frac{d_T}{b^2}\int_0^{t_T} \mathcal{L}_B^{-}(u)\,\mathcal{L}_B^{+}(u)\,K_{t_T}^{-}(u)\,K_{t_T}^{+}(u)\,du$$
$$= \frac{d_T G_b^0(t_T)}{b^2}\int_0^{t_T} \mathcal{L}_B^{-}(z)\,\mathcal{L}_B^{+}(z)\,f_{Z^0_{b/2,t_T,0}}(z)\,dz, \tag{S58}$$

where $f_{Z^0_{b/2,t_T,0}}(z)$ is the pdf of the beta random variable $Z^0_{b/2,t_T,0}$. By a similar argument as above, it can be shown that

$$
\begin{aligned}
&\mathcal{L}_B^-(z)\,\mathcal{L}_B^+(z)\\
&= \left[\left\{\frac{2t_T-1}{2t_T(1-t_T)}\right\} b - \left\{\frac{2t_T-1}{t_T^3(1-t_T)^3}\right\}\Delta^2\right](z-t_T)\\
&+ \left[\frac{1}{t_T^2(1-t_T)^2} + \left\{\frac{(2t_T-1)^2}{2t_T^3(1-t_T)^3}\right\} b - \left\{\frac{(2t_T-1)^2}{2t_T^4(1-t_T)^4}\right\}\Delta^2\right](z-t_T)^2\\
&- \frac{\Delta^2}{t_T^2(1-t_T)^2} + O(\Delta b)\,.
\end{aligned}
$$

Because

$$
\int_0^{t_T} f_{Z^0_{b/2,t_T,0}}(z)\,dz = \frac{1}{2} + O\left(b^{1/2}\right),
$$

$$
\int_0^{t_T} (z-t_T)\, f_{Z^\pm_{b/2,t_T,0}}(z)\,dz = \left(\frac{1-2t_T}{2}\right) b \left\{\frac{1}{2} + O\left(b^{1/2}\right)\right\},
$$

and

$$
\int_0^{t_T} (z-t_T)^2 f_{Z^\pm_{b/2,t_T,0}}(z)\,dz = \left\{\frac{t_T(1-t_T)}{2}\right\} b \left\{\frac{1}{2} + O\left(b^{1/2}\right)\right\},
$$

the integral part of (S58) can be simplified as

$$
\begin{aligned}
&\int_0^{t_T} \mathcal{L}_B^-(z)\,\mathcal{L}_B^+(z)\, f_{Z^0_{b/2,t_T,0}}(z)\,dz\\
&= \frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2} - \frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right) + o\left(\frac{\Delta^2}{b}\right)\right\} \qquad \text{(S59)}
\end{aligned}
$$

Substituting (S49) and (S59) into (S58), we can conclude that

$$
\begin{aligned}
V_{d_T}^{+-} &= \frac{d_T}{b^2}\left\{\frac{b^{-1/2}}{2\sqrt{\pi}\sqrt{t_T(1-t_T)}}\right\}\left\{1-\frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\
&\times\frac{b}{2t_T(1-t_T)}\left\{\frac{1}{2}-\frac{1}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\
&= \frac{b^{-3/2}d_T}{4\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{3/2}}\left\{\frac{1}{2}-\frac{3}{2t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\}. \quad \text{(S60)}
\end{aligned}
$$

(iii) Proof of (S44). Combining (S57) and (S60) leads to

$$
\begin{aligned}
V_{d_T} &= \left(V_{d_T}^{2-}+V_{d_T}^{2+}\right)-2V_{d_T}^{+-} \\
&= \frac{b^{-3/2}d_T}{4\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{3/2}}\left\{1+o\left(\frac{\Delta^2}{b}\right)\right\} \\
&- \frac{b^{-3/2}d_T}{4\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{3/2}}\left\{1-\frac{3}{t_T(1-t_T)}\left(\frac{\Delta^2}{b}\right)+o\left(\frac{\Delta^2}{b}\right)\right\} \\
&= \left(\frac{d_T}{2}\right)\frac{3}{2\sqrt{\pi}\left\{t_T(1-t_T)\right\}^{5/2}}\left(\frac{\Delta^2}{b^{5/2}}\right)+o\left(\frac{\Delta^2}{b}\right),
\end{aligned}
$$

which establishes (S44). This completes the proof. ■

S2.7.4. Proof of Lemma S7

It holds that

$$\begin{aligned}
&\left|\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}(\varsigma)-\left[-\sqrt{\frac{2}{\pi}}\frac{d_T}{\{t_T(1-t_T)\}^{3/2}}\right]\right| \\
&\leq \left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(\varsigma)-\hat{J}^{(2)}(t_T)\right|+\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(t_T)-E\left\{\hat{J}^{(2)}(t_T)\right\}\right| \\
&+\left|\left(\frac{b^{3/2}}{\Delta}\right)E\left\{\hat{J}^{(2)}(t_T)\right\}-\left[-\sqrt{\frac{2}{\pi}}\frac{d_T}{\{t_T(1-t_T)\}^{3/2}}\right]\right| \\
&=: D_1+D_2+D_3.
\end{aligned}$$

Then, the proof boils down to establishing the following three statements:

$$D_1=\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(\varsigma)-\hat{J}^{(2)}(t_T)\right|=o_p(1). \tag{S61}$$

$$D_2=\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(2)}(t_T)-E\left\{\hat{J}^{(2)}(t_T)\right\}\right|=o_p(1). \tag{S62}$$

$$D_3=\left|\left(\frac{b^{3/2}}{\Delta}\right)E\left\{\hat{J}^{(2)}(t_T)\right\}-\left[-\sqrt{\frac{2}{\pi}}\frac{d_T}{\{t_T(1-t_T)\}^{3/2}}\right]\right|=o(1). \tag{S63}$$

We work on (S63) first and then proceed to (S61) and (S62).

Proof of (S63). The proof is similar to that of Lemma S5, and thus its outline is presented. Because

$$E\left\{\ddot{K}_{t_T}^{\pm}(Y_i)\right\}=\int_0^1\ddot{K}_{t_T}^{\pm}(u)\,g_Y(u)\,du+d_T\int_0^{t_T}\ddot{K}_{t_T}^{\pm}(u)\,du=:I_{g_Y}^{\pm}+I_{d_T}^{\pm},$$

where

$$\ddot{K}_{t_T}^{\pm}(u) = \frac{1}{b^2}\left[\left\{\mathcal{L}_B^{\pm}(u)\right\}^2 - \left\{\Psi^{(1)}\left(p_{b,t_T,0}^{\pm}\right) + \Psi^{(1)}\left(q_{b,t_T}^{\pm}\right)\right\}\right] K_{t_T}^{\pm}(u),$$

it holds that

$$E\left\{\hat{J}^{(2)}(t_T)\right\} = E\left\{\ddot{K}_{t_T}^{-}(Y_i)\right\} - E\left\{\ddot{K}_{t_T}^{+}(Y_i)\right\} = \left(I_{g_Y}^{-} - I_{g_Y}^{+}\right) + \left(I_{d_T}^{-} - I_{d_T}^{+}\right).$$

By a similar procedure to the proof of Lemma S5, $I_{g_Y}^{\pm} = g_Y^{(2)}(t_T) \pm g_Y^{(3)}(t_T)\Delta + o(\Delta)$. Then, $\left|I_{g_Y}^{-} - I_{g_Y}^{+}\right|$ is at most $O(\Delta)$ because $\left|g_Y^{(3)}(t_T)\right| < \infty$ by Assumption 2. Following the steps in the proof of Lemma S5 also yields

$$\begin{aligned}
&\left\{\mathcal{L}_B^{\pm}(u)\right\}^2 - \left\{\Psi^{(1)}\left(p_{b,t_T,0}^{\pm}\right) + \Psi^{(1)}\left(q_{b,t_T}^{\pm}\right)\right\} \\
&= \left\{\mp 2\Delta + (2t_T - 1)b + O\left(\Delta^2\right)\right\}\left\{\frac{u - t_T}{t_T^2(1 - t_T)^2}\right\} \\
&+ \left[1 \mp \left\{\frac{2t_T - 1}{t_T(1 - t_T)}\right\}\Delta + \left\{\frac{(2t_T - 1)^2}{2t_T(1 - t_T)}\right\} b + O\left(\Delta^2\right)\right]\left\{\frac{(u - t_T)^2}{t_T^2(1 - t_T)^2}\right\} \\
&- \frac{b}{t_T(1 - t_T)} \mp \left\{\frac{2(2t_T - 1)}{t_T^2(1 - t_T)^2}\right\}\Delta b + O\left(\Delta^2\right).
\end{aligned}$$

Then, it can be found via (S40)-(S42) that

$$I_{d_T}^{\pm} = \frac{d_T}{b^2}\left\{\pm\frac{\Delta b^{1/2}}{\sqrt{2\pi}\left\{t_T(1 - t_T)\right\}^{3/2}} + o\left(\Delta b^{1/2}\right)\right\},$$

which implies that

$$I_{d_T}^{-} - I_{d_T}^{+} = -\sqrt{\frac{2}{\pi}} \frac{d_T}{\{t_T(1-t_T)\}^{3/2}} \left(\frac{\Delta}{b^{3/2}}\right) + o\left(\frac{\Delta}{b^{3/2}}\right).$$

Hence, (S63) is demonstrated.

Proof of (S61). By a mean-value expansion of $\hat{J}^{(2)}(\varsigma)$ around $\varsigma = t_T$,

$$\begin{aligned}\hat{J}^{(2)}(\varsigma) - \hat{J}^{(2)}(t_T) &= \hat{J}^{(3)}(\bar{\varsigma})(\varsigma - t_T) \\ &= \hat{J}^{(3)}(t_T)(\varsigma - t_T) + \left\{\hat{J}^{(3)}(\bar{\varsigma}) - \hat{J}^{(3)}(t_T)\right\}(\varsigma - t_T)\end{aligned}$$

for some $\bar{\varsigma}$ on the line segment joining ς and t_T. Then,

$$D_1 \leq \left(\frac{b^{3/2}}{\Delta}\right)\left\{\left|\hat{J}^{(3)}(t_T)\right| + \left|\hat{J}^{(3)}(\bar{\varsigma}) - \hat{J}^{(3)}(t_T)\right|\right\}|\varsigma - t_T|,$$

and it follows from Lipschitz continuity of $\hat{J}^{(3)}(\cdot)$ that $\left|\hat{J}^{(3)}(\bar{\varsigma}) - \hat{J}^{(3)}(t_T)\right|$ is of smaller order of magnitude than $\left|\hat{J}^{(3)}(t_T)\right|$. In short, $\left(b^{3/2}/\Delta\right)\left|\hat{J}^{(3)}(t_T)\right||\varsigma - t_T|$ is the dominant term in D_1.

In what follows, the order of magnitude in $\left|\hat{J}^{(3)}(t_T)\right|$ can be found via the identity

$$\hat{J}^{(3)}(t_T) \equiv E\left\{\hat{J}^{(3)}(t_T)\right\} + \left[\hat{J}^{(3)}(t_T) - E\left\{\hat{J}^{(3)}(t_T)\right\}\right].$$

Now,

$$\left|E\left\{\hat{J}^{(3)}(t_T)\right\}\right| \leq E\left|\dddot{K}^{-}_{t_T}(Y_i) - \dddot{K}^{+}_{t_T}(Y_i)\right|$$
$$\leq \int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| g_Y(u)\, du$$
$$+ |d_T| \int_0^{t_T} \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du,$$

where

$$\dddot{K}^{\pm}_{t_T}(u) = \frac{1}{b^3}\left[\left\{\mathcal{L}^{\pm}_B(u)\right\}^3 - 3\left\{\Psi^{(1)}\left(p^{\pm}_{b,t_T,0}\right) + \Psi^{(1)}\left(q^{\pm}_{b,t_T}\right)\right\}\mathcal{L}^{\pm}_B(u)\right.$$
$$\left.-\Psi^{(2)}\left(p^{\pm}_{b,t_T,0}\right) + \Psi^{(2)}\left(q^{\pm}_{b,t_T}\right)\right] K^{\pm}_{t_T}(u).$$

Because $g_Y(\cdot)$ is uniformly bounded on $[0,1]$ and $\int_0^{t_T}\left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du \leq \int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du$, we can see that the order of magnitude in $\left|E\left\{\hat{J}^{(3)}(t_T)\right\}\right|$ is determined by $\int_0^1 \left|\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)\right| du$. A mean-value expansion of $\dddot{K}^{\pm}_{t_T}(u)$ around $\Delta = 0$ implies that

$$\dddot{K}^{-}_{t_T}(u) - \dddot{K}^{+}_{t_T}(u)$$
$$\sim 2\left(\frac{\Delta}{b^4}\right)\left[-\left\{\mathcal{L}^0_B(u)\right\}^4 + 6\left\{\Psi^{(1)}\left(p^0_{b,t_T,0}\right) + \Psi^{(1)}\left(q^0_{b,t_T}\right)\right\}\left\{\mathcal{L}^0_B(u)\right\}^2\right.$$
$$+ 4\left\{\Psi^{(2)}\left(p^0_{b,t_T,0}\right) - \Psi^{(2)}\left(q^0_{b,t_T}\right)\right\}\mathcal{L}^0_B(u)$$
$$\left.-3\left\{\Psi^{(1)}\left(p^0_{b,t_T,0}\right) + \Psi^{(1)}\left(q^0_{b,t_T}\right)\right\}^2 + \left\{\Psi^{(3)}\left(p^0_{b,t_T,0}\right) + \Psi^{(3)}\left(q^0_{b,t_T}\right)\right\}\right] K_{B(t_T,b)}(u),$$

where $\mathcal{L}^0_B(u)$ is defined in (S34), and $K_{B(y,b)}(u)$ is Chen's (1999) original beta kernel. By extending Lemma A.2 of Funke and Hirukawa (2024) to higher-order moments of the log-transformed beta random variable, taking a similar approach to the proof of Lemma S6 and using (S7), it can be found that

$$\int_0^1 \left| \dddot{K}_{t_T}^{-}(u) - \dddot{K}_{t_T}^{+}(u) \right| du \leq O\left(\frac{\Delta}{b^4}\right) O\left(b^2\right) = O\left(\frac{\Delta}{b^2}\right),$$

Hence, $\left| E\left\{ \hat{J}^{(3)}(t_T) \right\} \right| \leq O\left(\Delta / b^2\right)$ is the case.

Furthermore,

$$\begin{aligned} & Var\left\{ \hat{J}^{(3)}(t_T) \right\} \\ & = \frac{1}{n}\left[E\left\{ \dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i) \right\}^2 - E^2\left\{ \dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i) \right\} \right], \end{aligned}$$

where $\left| E\left\{ \dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i) \right\} \right| = \left| E\left\{ \hat{J}^{(3)}(t_T) \right\} \right| = O\left(\Delta / b^2\right)$ as above. By taking into account that the order of magnitude in $E\left\{ \dddot{K}_{t_T}^{-}(Y_i) - \dddot{K}_{t_T}^{+}(Y_i) \right\}^2$ is determined by $\int_0^1 \left\{ \dddot{K}_{t_T}^{-}(u) - \dddot{K}_{t_T}^{+}(u) \right\}^2 du$ and again following a similar procedure to the proof of Lemma S6, it can be shown that

$$\int_0^1 \left\{ \dddot{K}_{t_T}^{-}(u) - \dddot{K}_{t_T}^{+}(u) \right\}^2 du \leq O\left(\frac{\Delta^2}{b^8}\right) O\left(b^{-1/2}\right) O\left(b^4\right) = O\left(\frac{\Delta^2}{b^{9/2}}\right).$$

It follows that $Var\left\{ \hat{J}^{(3)}(t_T) \right\} = O\left\{ \Delta^2 / \left(nb^{9/2}\right) \right\}$.

In conclusion,

$$\left| \hat{J}^{(3)}(t_T) \right| = O\left(\frac{\Delta}{b^2}\right) + O_p\left(\sqrt{\frac{\Delta^2}{nb^{9/2}}}\right).$$

Because ς lies between $\hat{t}_T$ and t_T and $\left|\hat{t}_T - t_T\right| = O_p\left(b^{1/2+\delta_1}\right) = o_p\left(b^{1/2}\right)$ as in the proof of Theorem 1, it holds that $|\varsigma - t_T| \leq \left|\hat{t}_T - t_T\right| = o_p\left(b^{1/2}\right)$. Therefore,

$$\left(\frac{b^{3/2}}{\Delta}\right)\left|\hat{J}^{(3)}\left(t_T\right)\right||\varsigma - t_T| = \left(\frac{b^{3/2}}{\Delta}\right)\left\{O\left(\frac{\Delta}{b^2}\right) + O_p\left(\sqrt{\frac{\Delta^2}{nb^{9/2}}}\right)\right\} o_p\left(b^{1/2}\right)$$
$$= o_p\left(1\right) + o_p\left(\frac{1}{\sqrt{nb^{1/2}}}\right) \xrightarrow{p} 0,$$

which establishes (S61).

Proof of (S62). It suffices to show that $Var\left\{\left(b^{3/2}/\Delta\right)\hat{J}^{(2)}\left(t_T\right)\right\} = o\left(1\right)$. To do so, again we focus on the order of magnitude in $E\left\{\ddot{K}^{-}_{t_T}\left(Y_i\right) - \ddot{K}^{+}_{t_T}\left(Y_i\right)\right\}^2$. Taking a mean-value expansion of $\ddot{K}^{\pm}_{t_T}\left(u\right)$ around $\Delta = 0$ and using the same notation as in the proof of (S61) above lead to

$$\ddot{K}^{-}_{t_T}\left(u\right) - \ddot{K}^{+}_{t_T}\left(u\right)$$
$$\sim 2\left(\frac{\Delta}{b^3}\right)\left[\left\{\mathcal{L}^0_B\left(u\right)\right\}^3 - 3\left\{\Psi^{(1)}\left(p^0_{b,t_T,0}\right) + \Psi^{(1)}\left(q^0_{b,t_T}\right)\right\}\mathcal{L}^0_B\left(u\right)\right.$$
$$\left.+\Psi^{(2)}\left(p^0_{b,t_T,0}\right) - \Psi^{(2)}\left(q^0_{b,t_T}\right)\right] K_{B(t_T,b)}\left(u\right).$$

Then, a similar argument to the proof of (S61) establishes that

$$E\left\{\ddot{K}^{-}_{t_T}\left(Y_i\right) - \ddot{K}^{+}_{t_T}\left(Y_i\right)\right\}^2 = O\left(\frac{\Delta^2}{b^6}\right) O\left(b^{-1/2}\right) O\left(b^3\right) = O\left(\frac{\Delta^2}{b^{7/2}}\right).$$

Because $Var\left\{\hat{J}^{(2)}\left(t_T\right)\right\} = O\left\{\Delta^2/\left(nb^{7/2}\right)\right\}$, it holds that

$$Var\left\{\left(\frac{b^{3/2}}{\Delta}\right)\hat{J}^{(2)}\left(t_T\right)\right\} = \left(\frac{b^3}{\Delta^2}\right)O\left(\frac{\Delta^2}{nb^{7/2}}\right) = O\left(\frac{1}{nb^{1/2}}\right) \to 0.$$

Therefore, (S62) is proven. This completes the proof. ■

S2.7.5. Proof of Lemma S8

Basically, a similar strategy to the proofs of (S61) and (S62) in Lemma S7 may be taken. The derivation is straightforward but much more tedious, and thus details are omitted. ■

S2.7.6. Proof of Theorem 2

In this proof, the limiting distribution of $\hat{t}_T$ in the transformed scale is first derived, and then the result is converted to the one in the original scale. Taking a mean-value expansion for the left-hand side of the first-order condition $\hat{J}^{(1)}\left(\hat{t}_T\right) = 0$, we have

$$\begin{aligned} 0 &= \hat{J}^{(1)}\left(t_T\right) + \hat{J}^{(2)}\left(\bar{t}_T\right)\left(\hat{t}_T - t_T\right) \\ &= E\left\{\hat{J}^{(1)}\left(t_T\right)\right\} + \left[\hat{J}^{(1)}\left(t_T\right) - E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}\right] + \hat{J}^{(2)}\left(\bar{t}_T\right)\left(\hat{t}_T - t_T\right) \quad \text{(S64)} \end{aligned}$$

for some $\bar{t}_T$ on the line segment joining $\hat{t}_T$ and t_T.

Rearranging (S64), we obtain

$$\sqrt{\frac{n}{b^{1/2}}}\left[\hat{t}_T - t_T - \left\{-\frac{E\left(\hat{J}^{(1)}\left(t_T\right)\right)}{\hat{J}^{(2)}\left(\bar{t}_T\right)}\right\}\right]$$
$$= -\sqrt{\frac{n}{b^{1/2}}}\left[\frac{\hat{J}^{(1)}\left(t_T\right) - E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}}{\hat{J}^{(2)}\left(\bar{t}_T\right)}\right]. \tag{S65}$$

Theorem 1 ensures that the hypothesis $\left|\hat{t}_T - t_T\right| = o_p\left(b^{1/2}\right)$ of Lemma S7 is satisfied. Then, in conjunction with Lemma S5, the bias term can be simplified as

$$-\frac{E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}}{\hat{J}^{(2)}\left(\bar{t}_T\right)} = \left(1 - \frac{t_T}{2}\right) b\left\{1 + o_p\left(1\right)\right\}.$$

We should also check Lyapunov's condition to demonstrate the asymptotic normality of $\sqrt{n/b^{1/2}}\left(b^{3/2}/\Delta\right)\left[\hat{J}^{(1)}\left(t_T\right) - E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}\right]$. This quantity can be expressed as

$$\sum_{i=1}^{n} R_i := \sum_{i=1}^{n} \sqrt{\frac{b^{5/2}}{n\Delta^2}}\left\{H_i - E\left(H_i\right)\right\}.$$

Then, by C_r-inequality, Jensen's inequality (due to the convexity of z^3 for $z \geq 0$) and Lemma S8,

$$E\left|R_i\right|^3 \leq 8\left(\frac{b^{5/2}}{n\Delta^2}\right)^{3/2} E\left|H_i\right|^3 = O\left(n^{-3/2} b^{-1/4}\right).$$

It also follows from Lemma S6 that $Var\left(R_i\right) = O\left(n^{-1}\right)$. Therefore,

$$\frac{\sum_{i=1}^{n} E\left|R_i\right|^3}{\left\{\sum_{i=1}^{n} Var\left(R_i\right)\right\}^{3/2}} = O\left(\frac{1}{\sqrt{nb^{1/2}}}\right) \to 0,$$

and Lyapunov's condition is indeed established.

By a central limit theorem, in conjunction with Lemma S6,

$$\sqrt{\frac{n}{b^{1/2}}}\left(\frac{b^{3/2}}{\Delta}\right)\left[\hat{J}^{(1)}\left(t_T\right) - E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}\right]$$
$$\xrightarrow{d} N\left(0, \frac{3}{2\sqrt{\pi}\left\{t_T\left(1-t_T\right)\right\}^{5/2}}\left\{\frac{f_Y(t_T^-) + f_Y(t_T^+)}{2}\right\}\right).$$

Finally, using Lemma S7 for the right-hand side of (S65) yields

$$-\sqrt{\frac{n}{b^{1/2}}}\left[\frac{\left(b^{3/2}/\Delta\right)\left\{\hat{J}^{(1)}\left(t_T\right) - E\left\{\hat{J}^{(1)}\left(t_T\right)\right\}\right\}}{\left(b^{3/2}/\Delta\right)\hat{J}^{(2)}\left(\bar{t}_T\right)}\right]$$
$$\xrightarrow{d} N\left(0, \frac{3\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}{4d_T^2}\left\{\frac{f_Y(t_T^-) + f_Y(t_T^+)}{2}\right\}\right).$$

So far we have obtained

$$\sqrt{\frac{n}{b^{1/2}}}\left[\hat{t}_T - t_T - \left(1 - \frac{t_T}{2}\right)b\left\{1 + o_p\left(1\right)\right\}\right]$$
$$\xrightarrow{d} N\left(0, \frac{3\sqrt{\pi}\sqrt{t_T\left(1-t_T\right)}}{4d_T^2}\left\{\frac{f_Y(t_T^-) + f_Y(t_T^+)}{2}\right\}\right). \qquad \text{(S66)}$$

By the definitions of $\hat{t}_B$ and t_T, the left-hand side of (S66) can be rewritten as

$$\sqrt{\frac{n}{b^{1/2}}}\left[T\left(\hat{t}_B\right)-T\left(t_0\right)-\left\{1-\frac{T\left(t_0\right)}{2}\right\}b\left\{1+o_p\left(1\right)\right\}\right].$$

Now, by a mean-value expansion, $T\left(\hat{t}_B\right)-T\left(t_0\right)=T^{(1)}\left(\check{t}\right)\left(\hat{t}_B-t_0\right)$ for some $\check{t}$ on the line segment joining $\hat{t}_B$ and t_0. By Assumption 3(ii), $T^{(1)}$ is Lipschitz continuous on $\mathbb{R}_+$, and thus we may also take $M>0$ as the Lipschitz constant. Because $\check{t}$ lies between $\hat{t}_B$ and t_0 and $\left|\hat{t}_B-t_0\right|=O_p\left(b^{1/2+\delta_1}\right)=o_p\left(b^{1/2}\right)$ by Theorem 1, it holds that $\left|\check{t}-t_0\right|\leq\left|\hat{t}_B-t_0\right|=o_p\left(b^{1/2}\right)$. Therefore,

$$\begin{aligned}
&\left|\left\{T\left(\hat{t}_B\right)-T\left(t_0\right)\right\}-T^{(1)}\left(t_0\right)\left(\hat{t}_B-t_0\right)\right|\\
&=\left|T^{(1)}\left(\check{t}\right)-T^{(1)}\left(t_0\right)\right|\left|\hat{t}_B-t_0\right|\\
&\leq M\left|\check{t}-t_0\right|\left|\hat{t}_B-t_0\right|\leq M\left|\hat{t}_B-t_0\right|^2=o_p\left(b\right).
\end{aligned}$$

In the end,

$$\begin{aligned}
&\sqrt{\frac{n}{b^{1/2}}}\left[T\left(\hat{t}_B\right)-T\left(t_0\right)-\left\{1-\frac{T\left(t_0\right)}{2}\right\}b\left\{1+o_p\left(1\right)\right\}\right]\\
&=\sqrt{\frac{n}{b^{1/2}}}\left[T^{(1)}\left(t_0\right)\left(\hat{t}_B-t_0\right)+o_p\left(b\right)-\left\{1-\frac{T\left(t_0\right)}{2}\right\}b\left\{1+o_p\left(1\right)\right\}\right]\\
&=\sqrt{\frac{n}{b^{1/2}}}T^{(1)}\left(t_0\right)\left[\hat{t}_B-t_0-\left\{\frac{1-T\left(t_0\right)/2}{T^{(1)}\left(t_0\right)}\right\}b\left\{1+o_p\left(1\right)\right\}\right]. \qquad \text{(S67)}
\end{aligned}$$

It also follows from

$$
\begin{aligned}
f_X(x) &= g_X(x) + d_0 \mathbf{1}\{x < t_0\} \\
&= g_Y\{T(x)\} T^{(1)}(x) + d_T T^{(1)}(x) \mathbf{1}\{x < t_0\}
\end{aligned}
$$

and $d_0 = d_T \, T^{(1)}(t_0)$ that

$$
\begin{aligned}
\frac{f_Y(t_T^-) + f_Y(t_T^+)}{2} &= g_Y(t_T) + \frac{d_T}{2} \\
&= \frac{1}{T^{(1)}(t_0)} \left\{ g_Y\{T(t_0)\} T^{(1)}(t_0) + \frac{d_T T^{(1)}(t_0)}{2} \right\} \\
&= \frac{1}{T^{(1)}(t_0)} \left\{ f_X(t_0^+) + \frac{d_0}{2} \right\} \\
&= \frac{1}{T^{(1)}(t_0)} \left\{ \frac{f_X(t_0^-) + f_X(t_0^+)}{2} \right\}.
\end{aligned}
$$

Then, by $d_0 = d_T \, T^{(1)}(t_0)$ and the definition of t_T, the asymptotic variance in (S66) reduces to

$$
\begin{aligned}
& \frac{3\sqrt{\pi}\sqrt{t_T(1 - t_T)}}{4 d_T^2} \left\{ \frac{f_Y(t_T^-) + f_Y(t_T^+)}{2} \right\} \\
&= \frac{3\sqrt{\pi}\sqrt{T(t_0)\{1 - T(t_0)\}}}{4\{d_0 / T^{(1)}(t_0)\}^2} \frac{1}{T^{(1)}(t_0)} \left\{ \frac{f_X(t_0^-) + f_X(t_0^+)}{2} \right\} \\
&= \frac{3\sqrt{\pi}\sqrt{T(t_0)\{1 - T(t_0)\}}}{4 d_0^2} T^{(1)}(t_0) \left\{ \frac{f_X(t_0^-) + f_X(t_0^+)}{2} \right\}. \qquad \text{(S68)}
\end{aligned}
$$

Substituting (S67) and (S68) into (S66) establishes the stated result. This completes the proof. ■

S2.8. Additional remarks on convergence results

S2.8.1. AMSE and super-consistency

Theorem 2 yields an approximation to the mean squared error (AMSE) of the splicing point estimator $\hat{t}_B$ as

$$\mathrm{AMSE}(\hat{t}_B) = \left\{\frac{1-T(t_0)/2}{T^{(1)}(t_0)}\right\}^2 b^2 + \frac{b^{1/2}}{n}V_B = O\left(b^2 + \frac{b^{1/2}}{n}\right).$$

Since both the $O(b^2)$ leading squared bias and the $O(b^{1/2}/n)$ leading variance terms vanish as $n \to \infty$ so that $b \to 0$, no bias-variance trade-off arises. In particular, the $b^{1/2}/n$ variance convergence rate exceeds the parametric $1/n$ rate, confirming super-consistency of $\hat{t}_B$.

S2.8.2. Efficiency comparison with the gamma kernel estimator

Readers may wonder whether estimating the splicing point in the original or transformed scale is more precise. For the splicing point estimator $\hat{t}_G$ in the original scale using the gamma kernel by Chen (2000), Theorem 2 of Funke and Hirukawa (2025) gives

$$\mathrm{Var}(\hat{t}_G) \sim \frac{b^{1/2}}{n}V_G := \frac{b^{1/2}}{n}\frac{3\sqrt{\pi}\,t_0^{1/2}}{4d_0^2}\left\{\frac{f_X(t_0^-)+f_X(t_0^+)}{2}\right\}.$$

Thus, $V_G/V_B = T^{(1)}(t_0)\sqrt{t_0/[T(t_0)\{1-T(t_0)\}]}$. Assuming $t_0 \approx t_M$ so that $T(t_0)\{1-T(t_0)\} \approx 1/4$, the ratio becomes approximately $2T^{(1)}(t_M)\sqrt{t_M}$, which can be simplified as

$$2T^{(1)}\left(t_M\right)\sqrt{t_M} = \begin{cases} 2/\left(\pi\sqrt{t_M}\right) & \text{for } T_1 \\ \ln 2/\sqrt{t_M} & \text{for } T_2 \\ 1/\left(2\sqrt{t_M}\right) & \text{for } T_3 \\ 3\ln 3/\left(4\sqrt{t_M}\right) & \text{for } T_4 \end{cases}$$

for transformations $T_1 - T_4$ in Table 1 of the main body. For each transformation, this quantity is less than one when $t_M \geq 1$, suggesting that $\hat{t}_B$ is asymptotically less efficient than $\hat{t}_G$. However, this comparison assumes a common bandwidth b for both estimators, which does not hold in practice. The efficiency of two estimators in finite-samples will be compared through the Monte Carlo study in Section S.3 shortly.

S2.8.3. Choice of b and Δ for super-consistency

For arbitrarily small $\delta_1, \delta_2 > 0$ as given in Assumption 4, put $\Delta \asymp b^{\alpha}$ with $\alpha \in (1/2 + \delta_1, 3/4)$ and $b \asymp n^{-\beta}$ with $\beta \in (0, (1-\delta_2)/(2\alpha - 1/2 + 4\delta_1))$. These choices jointly satisfy the shrinkage rates in Assumption 4. When $\alpha \in (1/2, 3/4)$, we have

$$\frac{1-\delta_2}{1+4\delta_1} < \frac{1-\delta_2}{2\alpha - 1/2 + 4\delta_1} < \frac{2(1-\delta_2)}{1+12\delta_1},$$

where the two bounds $\left(1-\delta_2\right)/\left(1+4\delta_1\right)$ and $2\left(1-\delta_2\right)/\left(1+12\delta_1\right)$ are slightly below 1 and 2, respectively. This yields the following conclusions:

1. We can always pick $\beta > 1/2$, and such β establishes that $\text{AMSE}(\hat{t}_B) = o(n^{-1})$, i.e., super-consistency of $\hat{t}_B$.
2. Setting $\beta = 2/3$ balances squared bias and variance so that $\text{AMSE}(\hat{t}_B) = O(n^{-4/3})$. When $nb^{3/2} \to 0$ (undersmoothing), the asymptotic normality in Theorem 2 reduces to $\sqrt{n/b^{1/2}}(\hat{t}_B - t_0) \xrightarrow{d} N(0, V_B)$.
3. The best achievable rate is $\text{AMSE}(\hat{t}_B) = O(n^{-2+\varepsilon})$ for arbitrarily small $\varepsilon > 0$.

It is worth emphasizing that for each of these choices, there is some $\kappa \in [0, 1)$ that satisfies $\ln n/(nb^{3/2-\kappa}) = O(1)$.

S3. Comprehensive simulation results

S3.1. Model specifications

Three alternative models are considered as the distribution of the univariate random variable $X \in \mathbb{R}_+$. Throughout, the splicing point is $t_0 = 4$. Characteristic numbers of these distributions are available in Table S1 below.

Model A (Log-Normal + Quadratic). The pdf is

$$f_X(x) = \left\{ \frac{1}{1 + (2/3)Dt_0} \right\} \left[\frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right\} + S(x) \right], \ (\mu, \sigma) = \left(\frac{3}{5}, \frac{1}{2} \right),$$

where

$$S(x) := D \left\{ 1 - \left(\frac{x - t_0}{t_0} \right)^2 \right\} \mathbf{1}\{x < t_0\},$$

and

$$d_0 = \; f_X\left(t_0^-\right) - f_X\left(t_0^+\right) = \frac{D}{1 + (2/3)\, D t_0}.$$

Putting $D = 3/52$ yields the jump size $d_0 = 0.05$.

Model B (Weibull & GPD). The pdf is

$$f_X(x) = f_L(x)\, \mathbf{1}\{x < t_0\} + (1 - c_L)\, f_R(x)\, \mathbf{1}\{x \geq t_0\},$$

where $c_L := \int_0^{t_0} f_X(x)\, dx = \int_0^{t_0} f_L(x)\, dx$ ensures unity of the integral of $f_X(x)$ over its entire support $\mathbb{R}_+$. The bulk model is the *Weibull* distribution with density

$$f_L(x) = \frac{\rho}{\lambda}\left(\frac{x}{\lambda}\right)^{\rho - 1} \exp\left\{-\left(\frac{x}{\lambda}\right)^{\rho}\right\},\ (\rho, \lambda) = \left(\frac{9}{4}, \frac{5}{2}\right).$$

The tail model is the *GPD* with density

$$f_R(x) = \frac{1}{s}\left\{1 + \frac{\xi(x - t_0)}{s}\right\}^{-(1+1/\xi)} \mathbf{1}\{x \geq t_0\}, (\xi, s) = \left(\frac{1}{4}, \frac{3}{2}\right).$$

The jump size is $d_0 = f_X\left(t_0^-\right) - f_X\left(t_0^+\right) \approx 0.05$.

Model C (Weibull & Half-Normal). The pdf and the bulk model are the same as those of Model B, whereas the tail model is the (shifted) *half-normal* distribution with density

$$f_R(x) = \frac{1}{\varpi}\sqrt{\frac{2}{\pi}} \exp\left\{-\frac{(x - t_0)^2}{2\varpi^2}\right\} \mathbf{1}\{x \geq t_0\}, \varpi = \frac{3}{\sqrt{2\pi}}.$$

The jump size is again $d_0 = f_X\left(t_0^-\right) - f_X\left(t_0^+\right) \approx 0.05$.

Table S1: Characteristic numbers of underlying distributions

Model	Distribution	Mode	c_L	$f_X(t_0^-)$	$f_X(t_0^+)$	d_0
A	Log-Normal + Quadratic	1.438	0.950	0.100	0.050	0.050
B	Weibull & GPD	1.925	0.944	0.091	0.038	0.054
C	Weibull & Half-Normal	1.925	0.944	0.091	0.038	0.054

S3.2. Competing automated threshold detection methods

The five automated threshold detection methods are:

(a) Minimum Kolmogorov-Smirnov distance procedure between empirical and GPD-based cdfs by Clauset et al. (2009) [KS].

(b) Minimum quantile discrepancy criterion for the mean absolute deviation between empirical and GPD-based quantiles by Danielsson et al. (2019) [Q-MAD].

(c) Minimum quantile discrepancy criterion for the sup-norm between empirical and GPD-based quantiles by Danielsson et al. (2019) [Q-SUP].

(d) Automated Eye-Balling method based on tail index estimates by Danielsson et al. (2019) [AEB].

(e) Anderson-Darling sequential testing procedure by Bader et al. (2018) [ADST].

   - Candidates are 20 empirical percentiles from 50.0% to 97.5% (increment 2.5%).

– The 5% significance level is used for testing.

– P-values for multiple tests are adjusted by the ForwardStop procedure.

R-Packages `poweRlaw`, `tea` and `eva` are employed to implement methods (a), (b)-(d) and (e), respectively.

S3.3. Implementation details of kernel estimators

S3.3.1. Shifted gamma estimator $\hat{t}_G$

The shifted gamma estimator $\hat{t}_G$ and its bias-corrected version $\tilde{t}_G = \hat{t}_G + b$ in the original scale by Funke and Hirukawa (2025) are implemented through the modified likelihood cross-validation (MLCV) [SG-ML, SG-ML-BC]. Candidates of b are taken from 100 equally-spaced grids over $[0.005, 0.500]$, and the exponent α for $\Delta = b^{\alpha}$ is chosen from $\{0.55, 0.60, 0.65, 0.70\}$.

S3.3.2. Shifted beta estimator $\hat{t}_B$

Let

$$\hat{f}^{\pm}_{Y,b,-i}(y;\alpha) := \frac{1}{n-1}\sum_{j=1,j\neq i}^{n} K^{\pm}_{y}(Y_j)$$

be density estimates using the sample with the ith observation eliminated. Also denote the number of observations falling into I_T as $n_0 := \sum_{i=1}^{n} \mathbf{1}\{Y_i \in I_T\}$. Using these notations, we focus on the least-squares cross-validation (LSCV) criterion $CV_{LS}(b;\alpha) = CV^{-}_{LS}(b;\alpha) + CV^{+}_{LS}(b;\alpha)$, where

$$CV^{\pm}_{LS}(b;\alpha) := \int_{I_T} \left\{\hat{f}^{\pm}_{Y,b}(y;\alpha)\right\}^2 dx - \frac{2}{n_0}\sum_{i:Y_i\in I_T} \hat{f}^{\pm}_{Y,b,-i}(Y_i;\alpha)$$

[SB-LS-$T_1 \sim T_4$]. Candidates of b are taken from 50 equally-spaced grids over $[0.005, 0.250]$, and the exponent α for $\Delta = b^\alpha$ is again chosen from $\{0.55, 0.60, 0.65, 0.70\}$. We have also investigated likelihood-based CV criteria by Marron (1985) and Van Es (1991), but we confirm superiority of LSCV.

The estimation procedure involves two optimizations, namely, (i) selecting (b, Δ) via grid search over CV criteria and (ii) maximizing $|\hat{J}(y)|$ on I_T through numerical optimization. Different algorithms are used because CV criteria are highly nonlinear (favoring grid search), while $|\hat{J}(y)|$ is concave on I_T (favoring numerical optimization).

S3.4. Comprehensive Monte Carlo results

Table S2 presents several performance measures, including the bias ("Bias"), standard deviation ("SD") and root mean squared error ("RMSE") of each threshold estimator over 1000 Monte Carlo samples. For kernel estimators, Monte Carlo averages and standard deviations (in parentheses) of CV smoothing parameters ("$\hat{b}$") are also reported for reference.

Table S2: Comprehensive Monte Carlo results

		$n = 250$					$n = 500$				
Estimator	α	Bias	SD	RMSE	b		Bias	SD	RMSE	b	
Model A: Log-Normal + Quadratic											
KS	–	−1.0402	0.5501	1.1767	–	(–)	−0.7721	0.4399	0.8886	–	(–)
Q-MAD	–	−0.2391	0.3979	0.4642	–	(–)	−0.1964	0.3963	0.4423	–	(–)
Q-SUP	–	0.1116	0.6432	0.6528	–	(–)	0.4523	0.8826	0.9918	–	(–)
AEB	–	0.5491	0.5357	0.7671	–	(–)	0.6200	0.3490	0.7115	–	(–)
ADST	–	−1.4147	0.7803	1.6156	–	(–)	−1.2834	0.7266	1.4748	–	(–)
Case (i): $I_0 = [3, 5]$											
SG-ML	0.70	−0.5966	0.4652	0.7565	0.0608	(0.0234)	−0.7394	0.4212	0.8510	0.0689	(0.0187)
SG-ML-BC	0.70	−0.5358	0.4520	0.7010	–	(–)	−0.6705	0.4104	0.7862	–	(–)
SB-LS-T_1	0.55	−0.5535	0.1371	0.5702	0.0397	(0.0064)	−0.5598	0.0923	0.5674	0.0398	(0.0046)
	0.60	−0.5276	0.1391	0.5456	0.0492	(0.0076)	−0.5319	0.0944	0.5402	0.0494	(0.0052)
	0.65	−0.5020	0.1400	0.5212	0.0590	(0.0086)	−0.5070	0.0951	0.5158	0.0592	(0.0061)
	0.70	−0.4794	0.1397	0.4994	0.0686	(0.0099)	−0.4837	0.0950	0.4929	0.0689	(0.0069)
SB-LS-T_2	0.55	−0.5954	0.1178	0.6070	0.0391	(0.0059)	−0.6013	0.0797	0.6066	0.0392	(0.0043)
	0.60	−0.5711	0.1208	0.5838	0.0487	(0.0069)	−0.5777	0.0816	0.5834	0.0487	(0.0049)
	0.65	−0.5489	0.1214	0.5622	0.0584	(0.0081)	−0.5537	0.0827	0.5598	0.0584	(0.0058)
	0.70	−0.5274	0.1227	0.5415	0.0680	(0.0093)	−0.5319	0.0832	0.5384	0.0681	(0.0066)
SB-LS-T_3	0.55	−0.4172	0.1671	0.4494	0.0233	(0.0049)	−0.4187	0.1176	0.4349	0.0233	(0.0036)
	0.60	−0.3857	0.1652	0.4196	0.0295	(0.0056)	−0.3859	0.1159	0.4029	0.0295	(0.0041)
	0.65	−0.3565	0.1631	0.3920	0.0358	(0.0064)	−0.3586	0.1153	0.3767	0.0358	(0.0046)
	0.70	−0.3323	0.1612	0.3693	0.0418	(0.0073)	−0.3315	0.1122	0.3499	0.0420	(0.0053)
SB-LS-T_4	0.55	−0.6742	0.0953	0.6809	0.0502	(0.0069)	−0.6811	0.0652	0.6842	0.0501	(0.0048)
	0.60	−0.6579	0.0991	0.6653	0.0620	(0.0080)	−0.6642	0.0667	0.6675	0.0620	(0.0057)
	0.65	−0.6411	0.1012	0.6491	0.0742	(0.0094)	−0.6466	0.0695	0.6503	0.0742	(0.0066)
	0.70	−0.6237	0.1037	0.6322	0.0864	(0.0108)	−0.6281	0.0702	0.6320	0.0865	(0.0076)
Case (ii): $I_0 = [3.5, 5.5]$											
SG-ML	0.70	−0.2460	0.3369	0.4172	0.0474	(0.0162)	−0.2921	0.3271	0.4385	0.0505	(0.0145)
SG-ML-BC	0.70	−0.1985	0.3302	0.3853	–	(–)	−0.2416	0.3186	0.3998	–	(–)
SB-LS-T_1	0.55	−0.3477	0.0981	0.3613	0.0524	(0.0072)	−0.3573	0.0715	0.3643	0.0524	(0.0051)
	0.60	−0.3364	0.1005	0.3511	0.0644	(0.0085)	−0.3442	0.0715	0.3516	0.0644	(0.0059)
	0.65	−0.3227	0.1029	0.3388	0.0768	(0.0099)	−0.3300	0.0740	0.3382	0.0767	(0.0068)
	0.70	−0.3076	0.1059	0.3253	0.0890	(0.0113)	−0.3143	0.0745	0.3230	0.0890	(0.0077)
SB-LS-T_2	0.55	−0.3620	0.0929	0.3737	0.0502	(0.0069)	−0.3714	0.0695	0.3779	0.0501	(0.0048)
	0.60	−0.3529	0.0955	0.3656	0.0619	(0.0081)	−0.3613	0.0695	0.3679	0.0618	(0.0056)
	0.65	−0.3405	0.0979	0.3542	0.0739	(0.0095)	−0.3478	0.0707	0.3549	0.0739	(0.0065)
	0.70	−0.3249	0.1006	0.3401	0.0861	(0.0110)	−0.3333	0.0719	0.3409	0.0859	(0.0076)
SB-LS-T_3	0.55	−0.2582	0.1220	0.2856	0.0320	(0.0057)	−0.2637	0.0862	0.2774	0.0321	(0.0038)
	0.60	−0.2371	0.1230	0.2671	0.0400	(0.0064)	−0.2428	0.0874	0.2581	0.0400	(0.0045)
	0.65	−0.2127	0.1214	0.2449	0.0482	(0.0075)	−0.2196	0.0856	0.2357	0.0481	(0.0051)
	0.70	−0.1907	0.1224	0.2266	0.0561	(0.0085)	−0.1944	0.0847	0.2121	0.0562	(0.0059)
SB-LS-T_4	0.55	−0.3995	0.0854	0.4085	0.0623	(0.0080)	−0.4115	0.0615	0.4161	0.0620	(0.0055)
	0.60	−0.3961	0.0854	0.4052	0.0767	(0.0095)	−0.4090	0.0631	0.4139	0.0763	(0.0065)
	0.65	−0.3910	0.0875	0.4007	0.0914	(0.0112)	−0.4030	0.0651	0.4082	0.0911	(0.0076)
	0.70	−0.3825	0.0910	0.3931	0.1064	(0.0129)	−0.3931	0.0670	0.3987	0.1061	(0.0090)
Case (iii): $I_0 = [2.5, 4.5]$											
SG-ML	0.70	−1.1608	0.4941	1.2616	0.1161	(0.1162)	−1.3476	0.3672	1.3967	0.1137	(0.0837)
SG-ML-BC	0.70	−1.0447	0.4801	1.1497	–	(–)	−1.2338	0.3618	1.2858	–	(–)
SB-LS-T_1	0.55	−0.6200	0.2251	0.6596	0.0293	(0.0053)	−0.6190	0.1597	0.6393	0.0294	(0.0039)
	0.60	−0.5842	0.2252	0.6261	0.0367	(0.0062)	−0.5821	0.1597	0.6036	0.0369	(0.0046)
	0.65	−0.5533	0.2231	0.5966	0.0443	(0.0073)	−0.5504	0.1577	0.5726	0.0446	(0.0052)
	0.70	−0.5269	0.2211	0.5714	0.0519	(0.0084)	−0.5243	0.1563	0.5471	0.0522	(0.0060)
SB-LS-T_2	0.55	−0.7289	0.1877	0.7526	0.0299	(0.0052)	−0.7310	0.1308	0.7426	0.0300	(0.0039)
	0.60	−0.6977	0.1885	0.7227	0.0374	(0.0062)	−0.6977	0.1316	0.7100	0.0376	(0.0044)
	0.65	−0.6683	0.1882	0.6943	0.0452	(0.0072)	−0.6691	0.1328	0.6822	0.0454	(0.0052)
	0.70	−0.6428	0.1877	0.6696	0.0529	(0.0083)	−0.6424	0.1316	0.6558	0.0532	(0.0059)
SB-LS-T_3	0.55	−0.4683	0.2400	0.5262	0.0171	(0.0038)	−0.4559	0.1724	0.4874	0.0171	(0.0029)
	0.60	−0.4327	0.2350	0.4923	0.0218	(0.0045)	−0.4206	0.1688	0.4532	0.0219	(0.0032)
	0.65	−0.4043	0.2311	0.4657	0.0267	(0.0051)	−0.3939	0.1657	0.4274	0.0268	(0.0037)
	0.70	−0.3807	0.2269	0.4432	0.0313	(0.0058)	−0.3706	0.1627	0.4047	0.0315	(0.0043)
SB-LS-T_4	0.55	−0.8635	0.1527	0.8769	0.0393	(0.0063)	−0.8707	0.1036	0.8768	0.0393	(0.0044)
	0.60	−0.8367	0.1551	0.8510	0.0489	(0.0075)	−0.8430	0.1067	0.8498	0.0490	(0.0052)
	0.65	−0.8114	0.1585	0.8268	0.0588	(0.0087)	−0.8158	0.1088	0.8230	0.0590	(0.0062)
	0.70	−0.7857	0.1590	0.8016	0.0687	(0.0100)	−0.7897	0.1094	0.7973	0.0690	(0.0071)

Table S2: *(continued)*

		$n = 250$					$n = 500$				
Estimator	α	Bias	SD	RMSE	b		Bias	SD	RMSE	b	
Model B: Splicing with Weibull & GPD											
KS	–	−1.1510	0.4794	1.2469	–	(–)	−1.0183	0.5832	1.1735	–	(–)
Q-MAD	–	−0.3241	0.3500	0.4771	–	(–)	−0.3154	0.3797	0.4936	–	(–)
Q-SUP	–	0.1315	0.9630	0.9719	–	(–)	0.5670	1.4624	1.5684	–	(–)
AEB	–	1.6324	1.1467	1.9949	–	(–)	1.7231	0.7848	1.8934	–	(–)
ADST	–	−1.0581	0.9258	1.4059	–	(–)	−0.7138	1.0491	1.2689	–	(–)
Case (i): $I_0 = [3, 5]$											
SG-ML	0.70	−0.5461	0.4892	0.7332	0.0484	(0.0202)	−0.5154	0.5629	0.7632	0.0499	(0.0197)
SG-ML-BC	0.70	−0.4976	0.4763	0.6888	–	(–)	−0.4655	0.5473	0.7185	–	(–)
SB-LS-T_1	0.55	−0.3760	0.1432	0.4023	0.0198	(0.0067)	−0.3604	0.1079	0.3762	0.0198	(0.0049)
	0.60	−0.3557	0.1378	0.3815	0.0256	(0.0080)	−0.3426	0.1036	0.3579	0.0257	(0.0057)
	0.65	−0.3392	0.1350	0.3651	0.0315	(0.0092)	−0.3268	0.1003	0.3418	0.0318	(0.0066)
	0.70	−0.3245	0.1340	0.3511	0.0373	(0.0105)	−0.3131	0.0990	0.3284	0.0376	(0.0076)
SB-LS-T_2	0.55	−0.4262	0.1281	0.4450	0.0207	(0.0064)	−0.4163	0.0960	0.4272	0.0206	(0.0047)
	0.60	−0.4089	0.1257	0.4278	0.0266	(0.0076)	−0.3990	0.0927	0.4097	0.0267	(0.0055)
	0.65	−0.3924	0.1255	0.4120	0.0328	(0.0089)	−0.3839	0.0900	0.3943	0.0330	(0.0064)
	0.70	−0.3794	0.1232	0.3989	0.0387	(0.0101)	−0.3709	0.0895	0.3816	0.0390	(0.0073)
SB-LS-T_3	0.55	−0.3114	0.1513	0.3462	0.0110	(0.0039)	−0.2948	0.1133	0.3158	0.0109	(0.0030)
	0.60	−0.2936	0.1431	0.3266	0.0142	(0.0047)	−0.2782	0.1091	0.2989	0.0143	(0.0035)
	0.65	−0.2759	0.1382	0.3086	0.0177	(0.0056)	−0.2623	0.1046	0.2824	0.0178	(0.0041)
	0.70	−0.2594	0.1370	0.2934	0.0210	(0.0065)	−0.2449	0.1007	0.2647	0.0212	(0.0047)
SB-LS-T_4	0.55	−0.5007	0.1194	0.5147	0.0294	(0.0080)	−0.4944	0.0861	0.5018	0.0294	(0.0060)
	0.60	−0.4862	0.1190	0.5005	0.0374	(0.0095)	−0.4795	0.0854	0.4871	0.0377	(0.0070)
	0.65	−0.4729	0.1178	0.4874	0.0457	(0.0111)	−0.4682	0.0840	0.4757	0.0459	(0.0080)
	0.70	−0.4606	0.1162	0.4750	0.0541	(0.0126)	−0.4561	0.0829	0.4636	0.0544	(0.0091)
Case (ii): $I_0 = [3.5, 5.5]$											
SG-ML	0.70	−0.2494	0.4820	0.5427	0.0374	(0.0122)	−0.2007	0.5973	0.6301	0.0367	(0.0132)
SG-ML-BC	0.70	−0.2120	0.4737	0.5190	–	(–)	−0.1639	0.5862	0.6087	–	(–)
SB-LS-T_1	0.55	−0.2452	0.1090	0.2684	0.0309	(0.0080)	−0.2419	0.0794	0.2546	0.0309	(0.0055)
	0.60	−0.2322	0.1058	0.2551	0.0398	(0.0096)	−0.2306	0.0771	0.2431	0.0397	(0.0066)
	0.65	−0.2181	0.1062	0.2426	0.0488	(0.0112)	−0.2154	0.0767	0.2286	0.0489	(0.0077)
	0.70	−0.2027	0.1057	0.2286	0.0578	(0.0129)	−0.2005	0.0764	0.2146	0.0579	(0.0088)
SB-LS-T_2	0.55	−0.2685	0.1021	0.2873	0.0311	(0.0076)	−0.2683	0.0748	0.2786	0.0310	(0.0053)
	0.60	−0.2572	0.1000	0.2760	0.0400	(0.0091)	−0.2564	0.0736	0.2667	0.0400	(0.0064)
	0.65	−0.2445	0.1004	0.2643	0.0490	(0.0108)	−0.2440	0.0718	0.2543	0.0490	(0.0074)
	0.70	−0.2299	0.1005	0.2509	0.0580	(0.0123)	−0.2290	0.0719	0.2400	0.0582	(0.0085)
SB-LS-T_3	0.55	−0.1759	0.1137	0.2095	0.0173	(0.0049)	−0.1716	0.0886	0.1931	0.0172	(0.0035)
	0.60	−0.1612	0.1111	0.1958	0.0226	(0.0059)	−0.1543	0.0839	0.1756	0.0226	(0.0042)
	0.65	−0.1404	0.1074	0.1768	0.0281	(0.0070)	−0.1338	0.0812	0.1565	0.0282	(0.0050)
	0.70	−0.1177	0.1082	0.1598	0.0335	(0.0082)	−0.1134	0.0784	0.1379	0.0336	(0.0057)
SB-LS-T_4	0.55	−0.3254	0.0954	0.3391	0.0422	(0.0094)	−0.3284	0.0687	0.3355	0.0422	(0.0065)
	0.60	−0.3191	0.0943	0.3327	0.0536	(0.0112)	−0.3202	0.0677	0.3272	0.0538	(0.0077)
	0.65	−0.3105	0.0947	0.3246	0.0655	(0.0132)	−0.3119	0.0680	0.3193	0.0656	(0.0091)
	0.70	−0.2999	0.0968	0.3152	0.0774	(0.0153)	−0.3017	0.0683	0.3094	0.0775	(0.0105)
Case (iii): $I_0 = [2.5, 4.5]$											
SG-ML	0.70	−0.7643	0.4375	0.8806	0.0982	(0.1093)	−0.7253	0.4562	0.8568	0.0878	(0.0745)
SG-ML-BC	0.70	−0.6661	0.4182	0.7865	–	(–)	−0.6375	0.4320	0.7701	–	(–)
SB-LS-T_1	0.55	−0.3657	0.2112	0.4223	0.0155	(0.0056)	−0.3459	0.1487	0.3766	0.0155	(0.0042)
	0.60	−0.3474	0.1995	0.4006	0.0199	(0.0067)	−0.3288	0.1439	0.3589	0.0200	(0.0050)
	0.65	−0.3343	0.1976	0.3883	0.0242	(0.0079)	−0.3134	0.1404	0.3433	0.0245	(0.0056)
	0.70	−0.3208	0.1942	0.3750	0.0285	(0.0089)	−0.2983	0.1375	0.3285	0.0291	(0.0063)
SB-LS-T_2	0.55	−0.4599	0.1915	0.4982	0.0161	(0.0058)	−0.4396	0.1350	0.4599	0.0161	(0.0043)
	0.60	−0.4425	0.1862	0.4801	0.0207	(0.0069)	−0.4220	0.1320	0.4422	0.0208	(0.0050)
	0.65	−0.4266	0.1823	0.4639	0.0254	(0.0081)	−0.4065	0.1298	0.4267	0.0257	(0.0059)
	0.70	−0.4132	0.1800	0.4507	0.0300	(0.0092)	−0.3924	0.1265	0.4123	0.0305	(0.0065)
SB-LS-T_3	0.55	−0.3197	0.2048	0.3796	0.0088	(0.0033)	−0.3064	0.1493	0.3408	0.0085	(0.0027)
	0.60	−0.3119	0.1977	0.3693	0.0110	(0.0040)	−0.2940	0.1450	0.3278	0.0111	(0.0032)
	0.65	−0.2998	0.1974	0.3590	0.0136	(0.0048)	−0.2802	0.1385	0.3126	0.0137	(0.0035)
	0.70	−0.2868	0.1918	0.3451	0.0160	(0.0056)	−0.2661	0.1380	0.2998	0.0163	(0.0040)
SB-LS-T_4	0.55	−0.5438	0.1827	0.5737	0.0232	(0.0077)	−0.5244	0.1280	0.5398	0.0232	(0.0057)
	0.60	−0.5251	0.1774	0.5543	0.0294	(0.0091)	−0.5057	0.1262	0.5212	0.0299	(0.0065)
	0.65	−0.5084	0.1733	0.5372	0.0361	(0.0104)	−0.4907	0.1242	0.5062	0.0365	(0.0074)
	0.70	−0.4933	0.1712	0.5221	0.0427	(0.0118)	−0.4768	0.1229	0.4924	0.0432	(0.0084)

Table S2: *(continued)*

		$n = 250$					$n = 500$				
Estimator	α	Bias	SD	RMSE	b		Bias	SD	RMSE	b	
					Model C: Splicing with Weibull & Half-Normal						
KS	–	−0.9789	0.3837	1.0514	–	(–)	−0.8630	0.2805	0.9074	–	(–)
Q-MAD	–	−0.3250	0.3040	0.4450	–	(–)	−0.3429	0.3064	0.4599	–	(–)
Q-SUP	–	0.3436	0.6683	0.7514	–	(–)	0.8676	0.6395	1.0779	–	(–)
AEB	–	0.5004	0.4249	0.6565	–	(–)	0.5785	0.2777	0.6417	–	(–)
ADST	–	−1.4826	0.5377	1.5771	–	(–)	−1.3992	0.6323	1.5354	–	(–)
					Case (i): $I_0 = [3, 5]$						
SG-ML	0.70	−0.6078	0.4844	0.7772	0.0562	(0.0243)	−0.5700	0.5669	0.8039	0.0577	(0.0234)
SG-ML-BC	0.70	−0.5515	0.4694	0.7242	–	(–)	−0.5123	0.5484	0.7504	–	(–)
SB-LS-T_1	0.55	−0.3317	0.1625	0.3694	0.0198	(0.0060)	−0.3158	0.1220	0.3385	0.0197	(0.0043)
	0.60	−0.3114	0.1540	0.3474	0.0257	(0.0071)	−0.2998	0.1148	0.3211	0.0256	(0.0052)
	0.65	−0.2942	0.1508	0.3306	0.0316	(0.0084)	−0.2827	0.1124	0.3042	0.0316	(0.0060)
	0.70	−0.2762	0.1494	0.3140	0.0374	(0.0098)	−0.2651	0.1091	0.2867	0.0376	(0.0069)
SB-LS-T_2	0.55	−0.3981	0.1409	0.4223	0.0212	(0.0058)	−0.3889	0.1050	0.4028	0.0211	(0.0043)
	0.60	−0.3805	0.1373	0.4045	0.0272	(0.0070)	−0.3719	0.0997	0.3851	0.0272	(0.0051)
	0.65	−0.3614	0.1343	0.3856	0.0335	(0.0082)	−0.3545	0.0974	0.3676	0.0336	(0.0059)
	0.70	−0.3442	0.1321	0.3687	0.0397	(0.0095)	−0.3379	0.0964	0.3514	0.0398	(0.0068)
SB-LS-T_3	0.55	−0.2661	0.1740	0.3180	0.0111	(0.0036)	−0.2531	0.1317	0.2853	0.0109	(0.0027)
	0.60	−0.2518	0.1625	0.2997	0.0143	(0.0043)	−0.2346	0.1212	0.2640	0.0144	(0.0032)
	0.65	−0.2302	0.1566	0.2784	0.0179	(0.0052)	−0.2169	0.1167	0.2463	0.0179	(0.0037)
	0.70	−0.2086	0.1554	0.2602	0.0214	(0.0061)	−0.1956	0.1136	0.2262	0.0214	(0.0043)
SB-LS-T_4	0.55	−0.4785	0.1279	0.4953	0.0302	(0.0075)	−0.4743	0.0919	0.4831	0.0302	(0.0054)
	0.60	−0.4637	0.1255	0.4804	0.0384	(0.0089)	−0.4594	0.0899	0.4682	0.0384	(0.0063)
	0.65	−0.4484	0.1242	0.4653	0.0469	(0.0102)	−0.4444	0.0877	0.4530	0.0470	(0.0074)
	0.70	−0.4343	0.1233	0.4514	0.0552	(0.0119)	−0.4305	0.0882	0.4394	0.0555	(0.0085)
					Case (ii): $I_0 = [3.5, 5.5]$						
SG-ML	0.70	−0.2616	0.5004	0.5647	0.0418	(0.0159)	−0.2291	0.6028	0.6448	0.0426	(0.0167)
SG-ML-BC	0.70	−0.2197	0.4905	0.5375	–	(–)	−0.1865	0.5895	0.6183	–	(–)
SB-LS-T_1	0.55	−0.1578	0.1200	0.1982	0.0328	(0.0085)	−0.1573	0.0860	0.1792	0.0325	(0.0061)
	0.60	−0.1422	0.1171	0.1842	0.0424	(0.0101)	−0.1443	0.0824	0.1662	0.0420	(0.0070)
	0.65	−0.1270	0.1164	0.1723	0.0520	(0.0118)	−0.1269	0.0817	0.1509	0.0518	(0.0083)
	0.70	−0.1089	0.1162	0.1593	0.0615	(0.0136)	−0.1084	0.0819	0.1358	0.0615	(0.0095)
SB-LS-T_2	0.55	−0.1902	0.1151	0.2223	0.0331	(0.0079)	−0.1953	0.0816	0.2117	0.0327	(0.0056)
	0.60	−0.1781	0.1119	0.2103	0.0426	(0.0095)	−0.1818	0.0785	0.1980	0.0423	(0.0066)
	0.65	−0.1627	0.1120	0.1975	0.0523	(0.0112)	−0.1639	0.0775	0.1813	0.0521	(0.0078)
	0.70	−0.1440	0.1116	0.1822	0.0620	(0.0128)	−0.1457	0.0772	0.1649	0.0618	(0.0090)
SB-LS-T_3	0.55	−0.0846	0.1269	0.1525	0.0183	(0.0050)	−0.0833	0.0948	0.1262	0.0180	(0.0037)
	0.60	−0.0696	0.1222	0.1406	0.0240	(0.0061)	−0.0664	0.0873	0.1096	0.0238	(0.0044)
	0.65	−0.0473	0.1216	0.1305	0.0298	(0.0073)	−0.0433	0.0854	0.0958	0.0298	(0.0052)
	0.70	−0.0219	0.1205	0.1224	0.0356	(0.0085)	−0.0198	0.0860	0.0882	0.0355	(0.0060)
SB-LS-T_4	0.55	−0.2515	0.1087	0.2740	0.0452	(0.0098)	−0.2578	0.0767	0.2690	0.0449	(0.0068)
	0.60	−0.2440	0.1079	0.2668	0.0575	(0.0118)	−0.2503	0.0736	0.2609	0.0571	(0.0082)
	0.65	−0.2332	0.1078	0.2569	0.0701	(0.0138)	−0.2376	0.0745	0.2490	0.0698	(0.0097)
	0.70	−0.2198	0.1086	0.2451	0.0828	(0.0160)	−0.2246	0.0749	0.2368	0.0825	(0.0111)
					Case (iii): $I_0 = [2.5, 4.5]$						
SG-ML	0.70	−0.8056	0.4221	0.9095	0.1275	(0.1394)	−0.7564	0.4502	0.8803	0.1080	(0.1005)
SG-ML-BC	0.70	−0.6781	0.4078	0.7913	–	(–)	−0.6484	0.4241	0.7748	–	(–)
SB-LS-T_1	0.55	−0.3191	0.2451	0.4024	0.0158	(0.0050)	−0.3003	0.1778	0.3490	0.0158	(0.0038)
	0.60	−0.3041	0.2323	0.3827	0.0201	(0.0059)	−0.2843	0.1668	0.3296	0.0204	(0.0043)
	0.65	−0.2905	0.2270	0.3686	0.0245	(0.0070)	−0.2676	0.1613	0.3124	0.0250	(0.0049)
	0.70	−0.2738	0.2234	0.3534	0.0288	(0.0080)	−0.2502	0.1588	0.2963	0.0295	(0.0056)
SB-LS-T_2	0.55	−0.4437	0.2115	0.4915	0.0169	(0.0054)	−0.4269	0.1517	0.4530	0.0170	(0.0041)
	0.60	−0.4260	0.2049	0.4728	0.0217	(0.0064)	−0.4080	0.1455	0.4331	0.0218	(0.0047)
	0.65	−0.4079	0.1996	0.4541	0.0264	(0.0074)	−0.3895	0.1422	0.4146	0.0268	(0.0053)
	0.70	−0.3916	0.1986	0.4391	0.0311	(0.0085)	−0.3729	0.1405	0.3985	0.0316	(0.0061)
SB-LS-T_3	0.55	−0.2820	0.2412	0.3711	0.0089	(0.0030)	−0.2646	0.1777	0.3187	0.0090	(0.0024)
	0.60	−0.2751	0.2299	0.3585	0.0114	(0.0037)	−0.2581	0.1684	0.3082	0.0115	(0.0029)
	0.65	−0.2591	0.2261	0.3439	0.0141	(0.0043)	−0.2391	0.1602	0.2878	0.0143	(0.0032)
	0.70	−0.2439	0.2214	0.3294	0.0165	(0.0050)	−0.2234	0.1601	0.2748	0.0169	(0.0037)
SB-LS-T_4	0.55	−0.5382	0.1971	0.5732	0.0242	(0.0072)	−0.5201	0.1395	0.5385	0.0245	(0.0054)
	0.60	−0.5173	0.1915	0.5516	0.0308	(0.0084)	−0.5002	0.1359	0.5184	0.0311	(0.0061)
	0.65	−0.4981	0.1878	0.5323	0.0376	(0.0097)	−0.4827	0.1340	0.5009	0.0380	(0.0070)
	0.70	−0.4807	0.1849	0.5150	0.0441	(0.0110)	−0.4657	0.1324	0.4841	0.0447	(0.0078)

S4. Details of empirical applications

S4.1. Data descriptions

(A) **Danish fire insurance losses**:

- Fire-related losses (in millions of Danish kroner) recorded between 1980 and 1990.
- Available as `danish` ($n = 2{,}492$) in R-package `SMPracticals`.

(B) **Norwegian fire insurance losses**:

- Variable `Loss2012` (in millions of 2012 Norwegian kroner) from `norfire` in R-package `CASdatasets`.
- A 1/3 subsample ($n = 3{,}064$) is randomly chosen from the full sample of 9,181 observations.

(C) **Belgian motor insurance losses**:

- Available as `beMTPL97` in `CASdatasets` (in thousands of Euros).
- A 1/6 subsample ($n = 3{,}151$) is randomly chosen from the full sample of 18,276 observations.

(D) **French motor insurance losses**:

- Available as `freMTPL2sev` in `CASdatasets` (in thousands of French francs).

- A 1/10 subsample (n = 2,638) is randomly chosen from the full sample of 26,444 observations.

Table S3: Descriptive statistics of datasets

Data	n	Mean	SD	SK	Min.	Q1	Q2	Q3	90%	95%	99%	Max.
(A) Danish Fire Insurance Losses (in millions of Danish kroner)												
Original	2,492	3.063	7.975	19.884	0.313	1.157	1.634	2.646	5.080	8.406	24.614	263.250
(B) Norwegian Fire Insurance Losses (in millions of Norwegian kroner)												
Original	9,181	5.235	18.277	26.287	0.741	1.526	2.354	4.096	8.762	15.414	50.783	881.448
Downsized	3,064	4.861	11.169	10.359	0.741	1.550	2.391	4.090	8.453	14.546	51.604	257.319
(C) Belgian Motor Insurance Losses (in thousands of Euros)												
Original	18,276	1.448	3.875	11.073	0.000	0.145	0.572	1.441	3.022	4.158	18.344	140.032
Downsized	3,151	1.451	3.833	9.009	0.001	0.145	0.529	1.426	3.021	3.957	20.451	81.946
(D) French Motor Insurance Losses (in thousands of French francs)												
Original	26,444	2.266	29.371	109.524	0.001	0.686	1.172	1.212	2.768	4.766	16.510	4,075.401
Downsized	2,638	2.608	27.168	42.476	0.001	0.739	1.172	1.304	2.912	5.369	18.551	1,301.173

Note:. n = sample size; Mean = average; SD = standard deviation; SK = skewness; Min. = minimum value; Q1 = first quartile; Q2 = median (i.e., second quartile); Q3 = third quartile; 90% = 90% quantile; 95% = 95% quantile; 99% = 99% quantile; and Max. = maximum value.

S4.2. ADST implementation details

- The 5% significance level and ForwardStop p-value adjustment are adopted.

- Threshold candidates are:

  (A) and (B) : 54 percentiles from 20.0% to 99.5% (increment 1.5%).

  (C) : 14 percentiles from 80.0% to 99.5% (increment 1.5%).

  (D) : 34 percentiles from 50.0% to 99.5% (increment 1.5%).

- If all candidates are rejected, then the 99.5% empirical percentile is selected as the threshold estimate by Bader et al. (2018).

S4.3. Comprehensive empirical results

Table S4: Comprehensive empirical results

Data	Estimator	Estimate of t_0	Estimator	α	I_0	Estimate of t_0	$\hat{b}$
		(A) Danish Fire Insurance Losses (in millions of Danish kroner)					
Original	KS	1.375	SG-ML	0.70	$[1, 30]$	1.861	0.235
	Q-MAD	29.037	SG-ML-BC	–	–	2.096	–
	Q-SUP	11.123					
	AEB	25.288	SB-LS-T$_3$	0.70	$[1, 30]$	1.808	0.005
	ADST	1.406					
		(B) Norwegian Fire Insurance Losses (in millions of Norwegian kroner)					
Downsized	KS	2.221	SG-ML	0.70	$[2, 50]$	2.756	0.350
	Q-MAD	35.794	SG-ML-BC	–	–	2.924	–
	Q-SUP	123.274					
	AEB	47.528	SB-LS-T$_3$	0.70	$[2, 50]$	2.702	0.005
	ADST	2.121					
		(C) Belgian Motor Insurance Losses (in thousands of Euros)					
Downsized	KS	3.233	SG-ML	0.70	$[2, 40]$	40.000‡	0.005
	Q-MAD	3.022	SG-ML-BC	–	–	40.005	–
	Q-SUP	40.035					
	AEB	18.592	SB-LS-T$_3$	0.70	$[2, 40]$	2.435	0.060
	ADST	29.451†					
		(D) French Motor Insurance Losses (in thousands of French francs)					
Downsized	KS	1.318	SG-ML	0.70	$[1, 20]$	20.000‡	0.005
	Q-MAD	3.204	SG-ML-BC	–	–	20.005	–
	Q-SUP	7.000					
	AEB	18.900	SB-LS-T$_3$	0.70	$[1, 20]$	11.247	0.005
	ADST	37.149†					

Note:. Superscripts "†" on ADST indicate that the 99.5% empirical percentile is chosen, whereas superscripts "‡" on SG-ML mean that an estimation failure occurs.

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